EECE 340-Signals and Systems
Homework #6

Problem 1

Determine the Z-transform for each of the following sequences:

a) $x(n) = e^n u(-n)$

b) $x(n) = e^n u(-n+1)$

c) $x(n) = a^n\left[u(n) - u(n-N)\right]$

Problem 2

Determine the inverse for each of the Z-transforms listed below using the inversion formula, long division and partial fraction expansion whenever applicable:

a) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \qquad |z| > \dfrac{1}{2}$

b) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \qquad |z| < \dfrac{1}{2}$

c) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

d) $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a}, \qquad |z| > \dfrac{1}{|a|}$

e) $X(z) = \dfrac{z^2}{(z-a)(z-b)}, \qquad |z| > \max(|a|,|b|)$

EECE 340 Homework #6 Solution

1)

a- $x(n) = e^{n} u(-n)$

$$X(z) = \sum_{n=-\infty}^{0} e^{n} z^{-n} = \sum_{n=0}^{\infty} (e^{-1} z)^{n}$$

$$= \frac{1}{1 - e^{-1} z} = \frac{-e}{z - e}, \quad |z| < e$$

b - $x(n) = e^{n} u(-n+1)$

$$X(z) = \sum_{n=-\infty}^{1} e^{n} z^{-n} = \sum_{n=-1}^{\infty} e^{-n} z^{n} = \sum_{n=-1}^{\infty} (e^{-1} z)^{n}$$

$$= \sum_{n=0}^{\infty} (e^{-1} z)^{n} + \frac{e}{z} = \frac{1}{1 - e^{-1} z} + \frac{e}{z}$$

$$= \frac{-e^{2}}{z(z-e)}, \quad 0 < |z| < e$$

c - $a^{n} [u(n) - u(n-N)]$

$$X(z) = \sum_{n=0}^{N-1} a^{n} z^{-n} = \sum_{n=0}^{N-1} (a z^{-1})^{n}$$

$$= \frac{1 - (a z^{-1})^{N}}{1 - a z^{-1}} = \frac{z^{N} - a^{N}}{z^{N-1}(z - a)}$$

$$= \frac{z^{N-1} + a z^{N-2} + \cdots + a^{N-1}}{z^{N-1}}, \quad |z| > 0$$

2)

a) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

Longdivision

$$
\begin{array}{r|l}
 & 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} + \cdots \\
\hline
1+\frac{1}{2}z^{-1} & 1 \\
 & 1 + \frac{1}{2}z^{-1} \\
\hline
 & -\frac{1}{2}z^{-1} \\
 & -\frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} \\
\hline
 & \frac{1}{4}z^{-2} \\
 & \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} \\
\hline
 & -\frac{1}{8}z^{-3}
\end{array}
$$

$$x(n) = \begin{cases} (-1/2)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Inverson formula

$X(z) = \dfrac{z}{z+\frac{1}{2}}, \quad |z| > \frac{1}{2}$

$$x(n) = \frac{1}{2\pi j}\oint_c \frac{z\, z^{n-1}}{(z+\frac{1}{2})}dz = \frac{1}{2\pi j}\oint_c \frac{z^n\, dz}{(z+\frac{1}{2})}$$

$$x(n) = z^n\Big|_{z=-\frac{1}{2}} = \left(-\tfrac{1}{2}\right)^n u(n).$$

b) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$

<u>Long division</u>

$$\begin{array}{r|l} & 2z - 4z^2 + 8z^3 - 16z^4 + 32z^5 - \cdots \\ \hline \frac{1}{2}z^{-1}+1 & 1 \\ & 1 + 2z \\ \hline & -2z \\ & -2z - 4z^2 \\ \hline & 4z^2 \\ & 4z^2 + 8z^3 \\ \hline & -8z^3 \\ & -8z^3 - 16z^4 \\ \hline & 16z^4 \end{array}$$

$$x(n) = \begin{cases} (-1)^{n-1}\, 2^{-n}, & n \le -1 \\ 0, & n \ge 0 \end{cases}$$

<u>Inversion formula</u>

$$X\left(\frac{1}{p}\right) = \frac{1}{1+\frac{1}{2}p} = \frac{2}{p+2}, \quad |p| > 2$$

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{2}{p+2} p^{-n-1} dp$$

$$= 2\left(p^{-n-1}\right)\Big|_{p=-2} = 2(-2)^{-n-1} u(-n-1)$$

$$= (-1)^{-n-1} 2^{-n} u(-n-1)$$

c) $X(z) = \dfrac{1 - \frac{1}{2} z^{-1}}{1 + \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}, \; |z| > \frac{1}{2}$

<u>Partial fraction</u>

1) $X(z) = \dfrac{8\left(1 - \frac{1}{2} z^{-1}\right)}{z^{-2} + 6 z^{-1} + 8} = \dfrac{-4 z^{-1} + 8}{(z^{-1}+4)(z^{-1}+2)}$

$$= \frac{A_1}{(z^{-1}+4)} + \frac{A_2}{(z^{-1}+2)}$$

$$A_1 = \frac{-4z^{-1}+8}{z^{-1}+2}\Bigg|_{z^{-1}=-4} = -12$$

$$A_2 = \frac{-4z^{-1}+8}{z^{-1}+4}\Bigg|_{z^{-1}=-2} = 8$$

$$X(z) = \frac{-12}{z^{-1}+4} + \frac{8}{z^{-1}+2}$$

$$= \frac{-3}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+\frac{1}{2}z^{-1}}$$

$$x(n) = -3\left(-\frac{1}{4}\right)^n u(n) + 4\left(-\frac{1}{2}\right)^n u(n)$$

2) $$X(z) = \frac{z(z-\frac{1}{2})}{z^2+\frac{3}{4}z+\frac{1}{8}} = \frac{z(z-\frac{1}{2})}{(z+\frac{1}{4})(z+\frac{1}{2})}$$

$$= 1 + \frac{-(5/4)z - \frac{1}{8}}{(z+\frac{1}{4})(z+\frac{1}{2})}$$

$$= 1 + \frac{A_1}{(z+\frac{1}{4})} + \frac{A_2}{(z+\frac{1}{2})}$$

$$A_1 = \left.\frac{-5/4\, z - 1/8}{(z+\frac{1}{2})}\right|_{z=-1/4} = 3/4$$

$$A_2 = \left.\frac{-5/4\, z - 1/8}{z+1/4}\right|_{z=-\frac{1}{2}} = -2$$

$$X(z) = 1 + \frac{3/4}{(z+\frac{1}{4})} + \frac{-2}{(z+\frac{1}{2})}$$

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$$X(z) = 1 + \frac{\frac{3}{4} z^{-1}}{1 + \frac{1}{4} z^{-1}} + \frac{-2\, z^{-1}}{1 + \frac{1}{2} z^{-1}}$$

$$x(n) = \delta(n) + \frac{3}{4}\left(-\frac{1}{4}\right)^{n-1} u(n-1) - 2\left(-\frac{1}{2}\right)^{n-1} u(n-1)$$

<u>Inversion formula</u>

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{z\left(z - \frac{1}{2}\right) z^{n-1}}{\left(z + \frac{1}{4}\right)\left(z + \frac{1}{2}\right)} dz$$

$$= \frac{1}{2\pi j} \oint_C \frac{\left(z - \frac{1}{2}\right) z^{n}}{\left(z + \frac{1}{4}\right)\left(z + \frac{1}{2}\right)} dz$$

$$= \left.\frac{\left(z - \frac{1}{2}\right) z^{n}}{\left(z + \frac{1}{4}\right)}\right|_{z = -\frac{1}{2}} + \left.\frac{\left(z - \frac{1}{2}\right) z^{n}}{\left(z + \frac{1}{2}\right)}\right|_{z = -1/4}$$

$$= 4\left(-\frac{1}{2}\right)^{n} u(n) - 3\left(-\frac{1}{4}\right)^{n} u(n).$$

d) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \quad |z| > \dfrac{1}{|a|}$

<u>long division</u>:

$$-\frac{1}{a} - \frac{1-a^2}{a^2}z^{-1} - \frac{(1-a^2)}{a^3}z^{-2} \ldots$$

$$-a + z^{-1} \,\big)\, 1 - az^{-1}$$

$$1 - \frac{1}{a}z^{-1}$$

$$\frac{1-a^2}{a}z^{-1}$$

$$\frac{1-a^2}{a}z^{-1} - \frac{1-a^2}{a^2}z^{-2}$$

$$+\frac{(1-a^2)}{a^2}z^{-2}$$

$$\frac{1-a^2}{a^2}z^{-2} - \frac{1-a^2}{a^3}z^{-3}$$

$$+\frac{(1-a^2)}{a^3}z^{-3}$$

$$x(n) = \begin{cases} 0, & n < 0 \\ -\frac{1}{a}, & n = 0 \\ -(1-a^2)\left(\frac{1}{a}\right)^{n+1}, & n \geq 1 \end{cases}$$

Inversion formula

$$X(z) = \frac{z-a}{1-az} = \frac{z-a}{-a(z-1/a)}, \quad |z| > \frac{1}{|a|}$$

$$x(n) = \frac{1}{2\pi j}\oint_c \frac{(z-a)z^{n-1}}{-a(z-1/a)}\,dz$$

$$= \frac{1}{2\pi j}\oint_c \frac{z^n}{-a(z-1/a)}\,dz + \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{(z-1/a)}\,dz$$

$$= -\frac{1}{a} z^n\Big|_{z=\frac{1}{a}} u(n) + z^{n-1}\Big|_{z=\frac{1}{a}} u(n-1)$$

$$= -\frac{1}{a}\left(\frac{1}{a}\right)^n u(n) + \left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= -\frac{1}{a}\delta(n) - \left(\frac{1}{a}\right)^{n+1} u(n-1) + \left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= -\frac{1}{a}\delta(n) - \left(\frac{1}{a}\right)^{n+1}(1-a^2)\,u(n-1)$$

Partial fraction

$$X(z) = \frac{1 - a z^{-1}}{z^{-1} - a} = \frac{1}{z^{-1} - a} - \frac{a z^{-1}}{z^{-1} - a}$$

$$= \frac{-1/a}{1 - 1/a\, z^{-1}} + \frac{z^{-1}}{1 - 1/a\, z^{-1}}$$

$$x(n) = \left(-\frac{1}{a}\right)\left(\frac{1}{a}\right)^{n} u(n) + \left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= -\frac{1}{a}\,\delta(n) - \left(\frac{1}{a}\right)^{n+1}(1 - a^2)\, u(n-1).$$

e) $X(z) = \dfrac{z^2}{(z-a)(z-b)}$, $|z| > \max(|a|, |b|)$

partial fraction

$$X(z) = \frac{z^2}{z^2 - (a+b)z + ab}$$

$$= 1 + \frac{(a+b)z - ab}{(z-a)(z-b)}$$

$$= 1 + \frac{A_1}{(z-a)} + \frac{A_2}{(z-b)}$$

$$A_1 = \left.\frac{(a+b)z - ab}{(z-b)}\right|_{z=a} = \frac{a^2}{(a-b)}$$

$$A_2 = \left.\frac{(a+b)z - ab}{(z-a)}\right|_{z=b} = \frac{b^2}{(b-a)}$$

$$X(z) = 1 + \frac{a^2}{(a-b)(z-a)} + \frac{b^2}{(b-a)(z-b)}$$

$$= 1 + \frac{a^2}{a-b}\,\frac{z^{-1}}{1-az^{-1}} + \frac{b^2}{(b-a)}\cdot\frac{z^{-1}}{1-bz^{-1}}$$

$$x(n) = \delta(n) + \frac{a^2}{a-b}\,a^{n-1}u(n-1) + \frac{b^2}{b-a}\,b^{n-1}u(n-1)$$

$$= \frac{a}{(a-b)}\,a^n u(n) + \frac{b}{b-a}\,b^n u(n)$$

<u>Inversion formula</u>

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^2 z^{n-1}}{(z-a)(z-b)}\,dz$$

$$= \frac{1}{2\pi j}\oint_C \frac{z^{n+1}}{(z-a)(z-b)}\,dz$$

$$x(n) = \frac{z^{n+1}}{(z-a)}\Big|_{z=b} + \frac{z^{n+1}}{(z-b)}\Big|_{z=a}$$

$$= \frac{b^{n+1}}{(b-a)} u(n+1) + \frac{a^{n+1}}{a-b} u(n+1)$$

$x(n) = 0$, for $n = -1 \Rightarrow$

$$x(n) = \frac{b^{n+1}}{(b-a)} u(n) + \frac{a^{n+1}}{a-b} u(n).$$

$$= \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right] u(n).$$

Long division

$$X(z) = \frac{z^2}{(z-a)(z-b)}$$

$$= \frac{1}{(1-az^{-1})(1-bz^{-1})}$$

$$= \frac{1}{1-(a+b)z^{-1}+abz^{-2}}$$

$$1-(a+b)z^{-1}+abz^{-2} \;\Big)\; 1 \qquad \text{quotient: } 1+(a+b)z^{-1}+(a^2+b^2+ab)z^{-2}+(a+b)(a^2+b^2)z^{-3}+\cdots$$

$$1$$

$$1-(a+b)z^{-1}+abz^{-2}$$

$$(a+b)z^{-1}-abz^{-2}$$

$$(a+b)z^{-1}-(a+b)^2z^{-2}+(a+b)abz^{-3}$$

$$(a^2+b^2+ab)z^{-2}-(a+b)abz^{-3}$$

$$(a^2+b^2+ab)z^{-2}-(a+b)(a^2+b^2+ab)z^{-3}+ab(a^2+b^2+ab)z^{-4}$$

$$(a+b)(a^2+b^2)z^{-3}-ab(a^2+b^2+ab)z^{-4}$$

$$x(n)=\frac{1}{a-b}\left[a^{n+1}-b^{n+1}\right]u(n)$$