EECE 340-Signals and Systems
Homework #7

Problem # 1

Consider the following discrete-time sequence:

$$h(n) = Ar^n \cos(n\omega_0 T + \theta)\, u(n),$$

where $0<r<1$.

a) Find H (z) and specify its region of convergence

b) Let $\omega_0 = \frac{\pi}{T}$ and plot the pole-zero diagram for H (z).

Problem # 2

Show the following:

a) The Z-transform of $x^*(n)$ is $X^*(z^*)$

b) The Z-transform of $x(-n)$ is $X(1/z)$

c) The Z-transform of $a^n x\ (n)$ is $X(z/a)$

d) The Z-transform of $nx(n)$ is $-z\frac{dX(z)}{dz}$

e) The initial value $x(0)$ of a causal sequence $x(n)$ is given by $\lim_{z\to\infty} X(z)$

f) The initial value $x(0)$ for a sequence $x(n)$ such that $x(n)=0,\ n>0$ is given by $\lim_{z\to 0} X(z)$.

In part (c), show how the poles and zeros of X (z) are moved.

EECE 340 Homework 7 Solution

Problem # 1

a) $$H(z) = \frac{A}{2}\sum_{n=0}^{\infty} r^n\left[e^{j(n\omega_0T+\theta)} + e^{-j(n\omega_0T+\theta)}\right]z^{-n}$$

$$= \frac{A}{2}\left\{e^{j\theta}\sum_{n=0}^{\infty}\left(re^{j\omega_0T}z^{-1}\right)^n + e^{-j\theta}\sum_{n=0}^{\infty}\left(re^{-j\omega_0T}z^{-1}\right)^n\right\}$$

$$= \frac{A}{2}\left\{\frac{e^{j\theta}}{1-re^{j\omega_0T}z^{-1}} + \frac{e^{-j\theta}}{1-re^{-j\omega_0T}z^{-1}}\right\}$$

$$= \frac{A}{2}\left\{\frac{e^{j\theta}-re^{-j(\omega_0T-\theta)}z^{-1}+e^{-j\theta}-re^{j(\omega_0T-\theta)}z^{-1}}{\left(1-re^{j\omega_0T}z^{-1}\right)\left(1-re^{-j\omega_0T}z^{-1}\right)}\right\}$$

$$= A\left\{\frac{\cos\theta - rz^{-1}\cos(\omega_0T-\theta)}{\left(1-re^{j\omega_0T}z^{-1}\right)\left(1-re^{-j\omega_0T}z^{-1}\right)}\right\}$$

$$= A\left\{\frac{z^2\left(\cos\theta - rz^{-1}\cos(\omega_0T-\theta)\right)}{\left(z-re^{j\omega_0T}\right)\left(z-re^{-j\omega_0T}\right)}\right\}$$

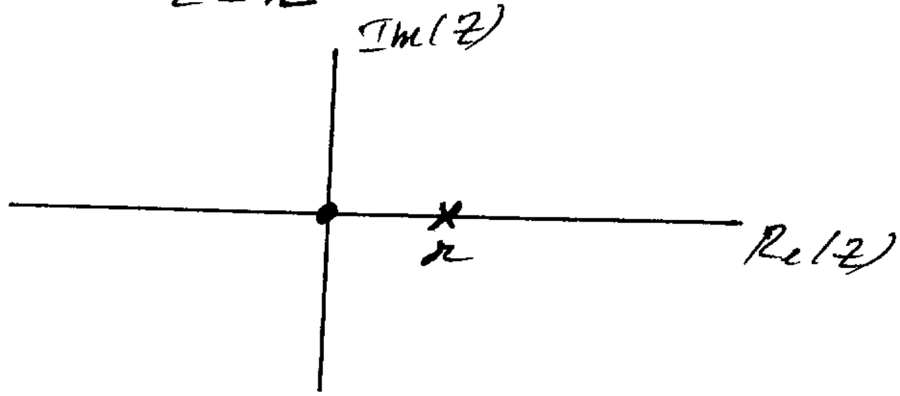
$$= A\cos\theta\left\{\frac{z\left(z-r\frac{\cos(\omega_0T-\theta)}{\cos\theta}\right)}{\left(z-re^{j\omega_0T}\right)\left(z-re^{-j\omega_0T}\right)}\right\}$$

$|z|>r$

b) $\omega_0 = \frac{\pi}{T} \Rightarrow$

$$H(z) = A\cos\theta \left\{ \frac{z(z+r)}{(z - re^{j\eta})(z + re^{-j\eta})} \right\}$$

$$= A\cos\theta \, \frac{z}{z-r}$$



z-plane

Problem # 2

a) $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z^*) = \sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}$$

$$X^*(z^*) = \left\{ \sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n} \right\}^* = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \mathcal{Z}\{x^*(n)\}$$

b) $$X\left(\tfrac{1}{z}\right) = \sum_{n=-\infty}^{\infty} x(n) z^{n} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \mathcal{Z}[x(-n)].$$

c) $X\left(\frac{z}{a}\right) = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$

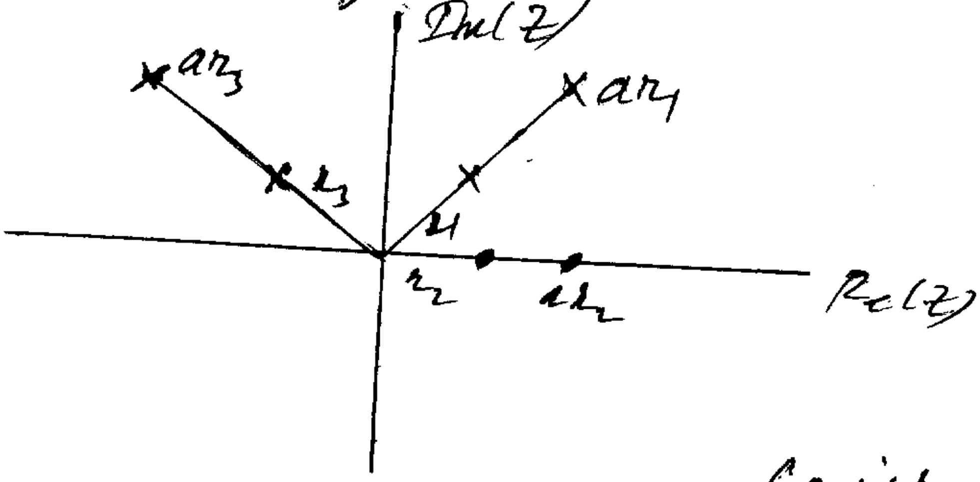
$= \mathcal{Z}\left[a^n x(n)\right].$

If $X(z)$ has a pole or zero at $z = z_1 = r_1 e^{j\omega_1}$, then $X(z/a)$ has a pole or zero at $z = az_1 = ar_1 e^{j\omega_1}$

$\Rightarrow z = |a| e^{j\angle a} r_1 e^{j\omega_1}$

$\Rightarrow |z| = |a| r_1 \quad , \quad \angle z = \angle a + \omega_1$

Thus, the poles and zeros of $X(z)$ are scaled by $|a|$ and rotated by $\underline{\angle a}$.



z-plane

(a is real > 0)

d) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)(-n z^{-n-1}) = -\frac{1}{z}\sum_{n=-\infty}^{\infty} n x(n) z^{-n}$

$$\Rightarrow \sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n} = -z\,\frac{dX(z)}{dz} = Z[n\,x(n)].$$

e) $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

$$\lim_{z\to\infty} X(z) = \sum_{n=-\infty}^{\infty} x(n) \lim_{z\to\infty} z^{-n}$$

But $$\lim_{z\to\infty} z^{-n} = \begin{cases} 0, & n>0 \\ 1, & n=0 \end{cases}$$

$$\Rightarrow \lim_{z\to\infty} X(z) = x(0)$$

f) $$X(z) = \sum_{n=-\infty}^{0} x(n)\,z^{-n}$$

$$\lim_{z\to 0} X(z) = \sum_{n=-\infty}^{0} x(n) \lim_{z\to 0} z^{-n}$$

But $$\lim_{z\to 0} z^{-n} = \begin{cases} 0, & n<0 \\ 1, & n=0 \end{cases}$$

$$\Rightarrow \lim_{z\to 0} X(z) = x(0)$$