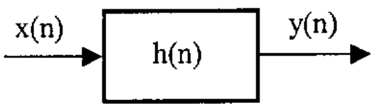
EECE 340-Signals and Systems
Homework #8

Problem # 1

The system shown below is LTI. Let the system input and impulse response be respectively given by:

$$x(n) = u(n) - u(n-N)$$

$$h(n) = a^n u(n)$$



Determine the system output, $y(n)$, using the inverse Z-transform.

Problem # 2

A causal, linear and shift-invariant discrete-time system is described by the following difference equation:

$$y(n) = y(n-1) + y(n-2) + x(n-1).$$

a) Find H (z), plot the poles and zeros of H (z) and indicate the ROC.
b) Find the unit sample (impulse) response of the system.
c) Find a stable and non-causal unit sample response that satisfies the given difference equation.

Problem # 3

Consider a causal, linear and shift-invariant discrete-time system with transfer function, H(z), given by

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}}, \text{ where } a \text{ is real.}$$

a) For what range of values of a the system is stable.
b) For $0 < a < 1$ plot the pole-zero diagram and specify the ROC.
c) Show graphically in the z-plane that this system is an all pass; i.e., $|H(\omega)| =$ constant for all ω.

Problem # 4

Consider the following difference equation representing a causal discrete-time system:

$$y(n) - 2r\cos\theta\, y(n-1) + r^2 y(n-2) = x(n).$$

Determine the system output, $y(n)$, if its input is given by $x(n) = \alpha^n u(n)$.

Problem # 5

A discrete-time system has the following transfer function:

$$H(z) = \frac{1 - z^{-1}}{1 - 2z^{-1} + 0.75z^{-2}}$$

a) Determine the impulse response, $h(n)$, of the system if it is assumed causal.
b) Determine the impulse response, $h(n)$, of the system if it is assumed stable.

1) $h(n) = a^n u(n)$, $x(n) = u(n) - u(n-N)$
$= x_1(n) - x_2(n)$

$$H(z) = \frac{1}{1 - az^{-1}}, \ |z| > |a|$$

$$X_1(z) = \frac{1}{1 - z^{-1}}, \ |z| > 1$$

$$X_2(z) = \frac{z^{-N}}{1 - z^{-1}}, \ |z| > 1$$

$$Y_1(z) = H(z)X_1(z) = \frac{1}{(1-z^{-1})(1-az^{-1})}$$

$$= \frac{z^2}{(z-1)(z-a)}, |z| > \max(|a|, 1)$$

$$Y_2(z) = H(z)X_2(z) = \frac{z^{-N}}{(1-z^{-1})(1-az^{-1})}$$

$$= \frac{z^{-N+2}}{(z-1)(z-a)}, \ |z| > \max(|a|, 1)$$

$$y_1(n) = \frac{1}{2\pi j} \oint_C \frac{z^2 z^{n-1}}{(z-1)(z-a)} dz$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{(z-1)(z-a)} dz$$

$$= \frac{z^{n+1}}{(z-a)}\bigg|_{z=1} + \frac{z^{n+1}}{(z-1)}\bigg|_{z=a}$$

$$= \frac{a^{n+1} - 1}{a-1}, \quad n \geq 0$$

$$y_2(n) = \frac{1}{2\pi j} \oint_C \frac{z^{-N+2} z^{n-1}}{(z-1)(z-a)} dz$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^{n-N+1}}{(z-1)(z-a)} dz$$

$$= \frac{z^{n-N+1}}{z-a}\bigg|_{z=1} + \frac{z^{n-N+1}}{(z-1)}\bigg|_{z=a}$$

$$= \frac{a^{n-N+1} - 1}{a-1}, \quad n \geq N-1$$

Thus, $y(n) = y_1(n) - y_2(n)$ becomes

$$y(n) = \frac{a^{n+1}-1}{a-1} - \frac{a^{n-N+1}-1}{a-1}$$

$$= \frac{a^{n+1} - a^{n-N+1}}{a-1} = a^{n+1}\left[\frac{1-a^{-N}}{a-1}\right], \quad n \geq N$$

$$y(n) = \frac{a^{n+1}-1}{a-1}, \quad 0 \leq n \leq N-1$$

2) $y(n) = y(n-1) + y(n-2) + x(n-1)$

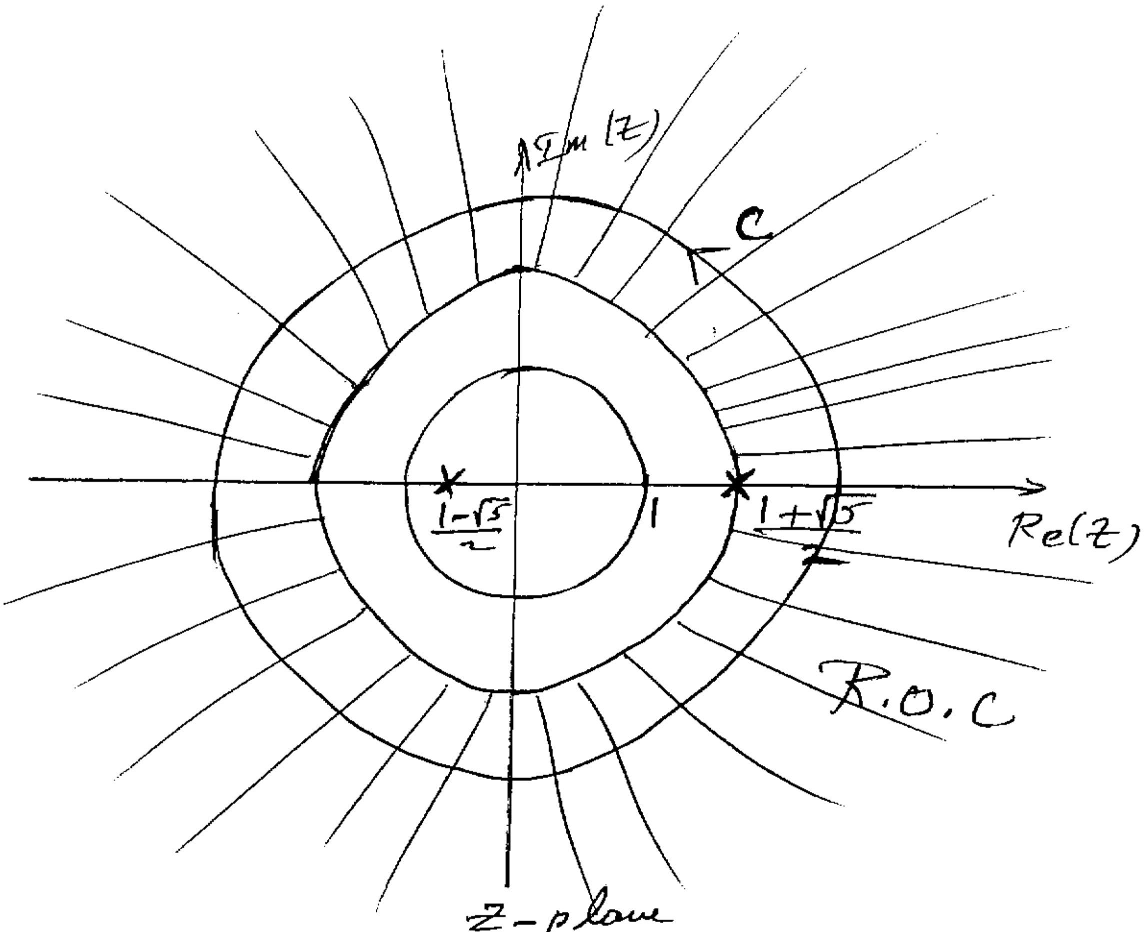
a) $Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$

$$Y(z)\left[1 - z^{-1} - z^{-2}\right] = z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

$$= \frac{z}{z^2 - z - 1}$$

$$= \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}, \quad |z| > \frac{1+\sqrt{5}}{2}$$



b)

$$h(n) = \frac{1}{2\pi j}\oint_C \frac{z^n}{\left(z-\frac{1+\sqrt{5}}{2}\right)\left(z-\frac{1-\sqrt{5}}{2}\right)}dz$$

$$= \left.\frac{z^n}{\left(z-\frac{1+\sqrt{5}}{2}\right)}\right|_{z=\frac{1-\sqrt{5}}{2}} + \left.\frac{z^n}{\left(z-\frac{1-\sqrt{5}}{2}\right)}\right|_{z=\frac{1+\sqrt{5}}{2}}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0$$

$$= 0 \qquad , \quad n < 0$$

c) A stable (non-causal) filter is one whose R.O.C. includes the unit circle. Thus,

$$H(z) = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}, \quad \frac{1-\sqrt{5}}{2} < |z| < \frac{1+\sqrt{5}}{2}$$

$$h(n) = \frac{1}{2\pi j} \oint_C \frac{z^n}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} dz$$

where, C is located in the above indicated R.O.C.

$$h(n) = \left.\frac{z^n}{\left(z - \frac{1+\sqrt{5}}{2}\right)}\right|_{z = \frac{1-\sqrt{5}}{2}} = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geq 0$$

$$H(p^{-1}) = \frac{-p}{p^2 + p - 1}$$

$$= \frac{-p}{\left(p - \frac{-1+\sqrt{5}}{2}\right)\left(p - \frac{-1-\sqrt{5}}{2}\right)},$$

$$\frac{-1+\sqrt{5}}{2} < |p| < \left|\frac{-1-\sqrt{5}}{2}\right|$$

$$h(n) = \frac{1}{2\pi j} \oint_{c'} \frac{-\,p^{-n}}{\left(p - \frac{-1+\sqrt{5}}{2}\right)\left(p - \frac{-1-\sqrt{5}}{2}\right)} \, dp$$

where, c' is located in the above region of convergence.

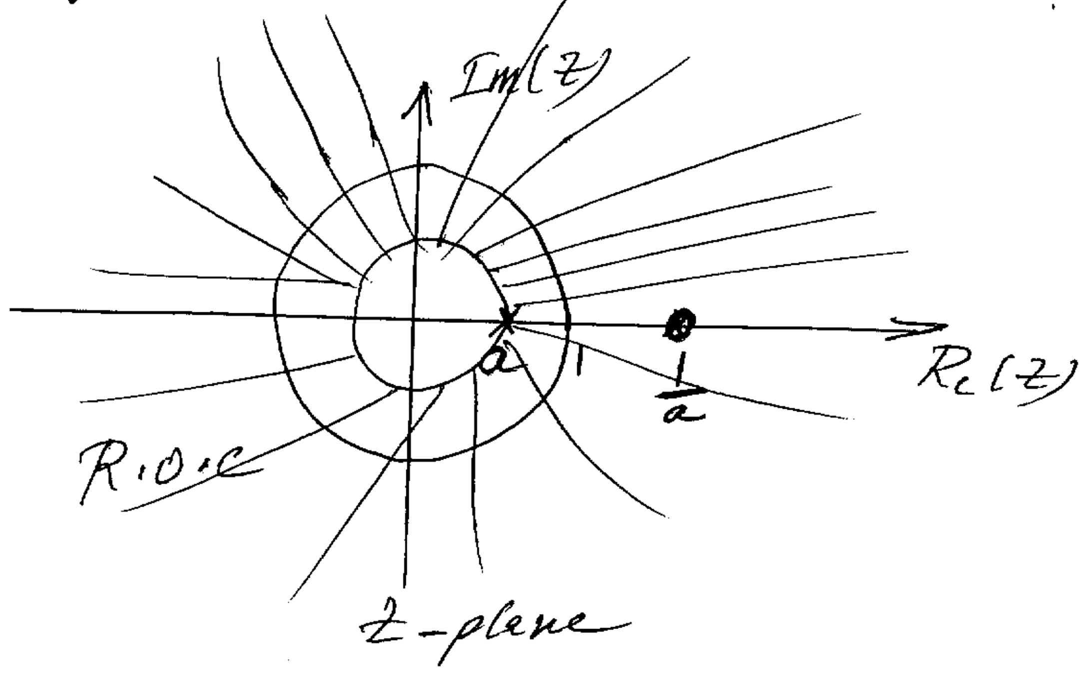
$$h(n) = \left. \frac{-\,p^{-n}}{\left(p - \frac{-1-\sqrt{5}}{2}\right)} \right|_{p = \frac{-1+\sqrt{5}}{2}}$$

$$= -\frac{1}{\sqrt{5}} \left(\frac{-1+\sqrt{5}}{2}\right)^{-n}, \quad n < 0.$$

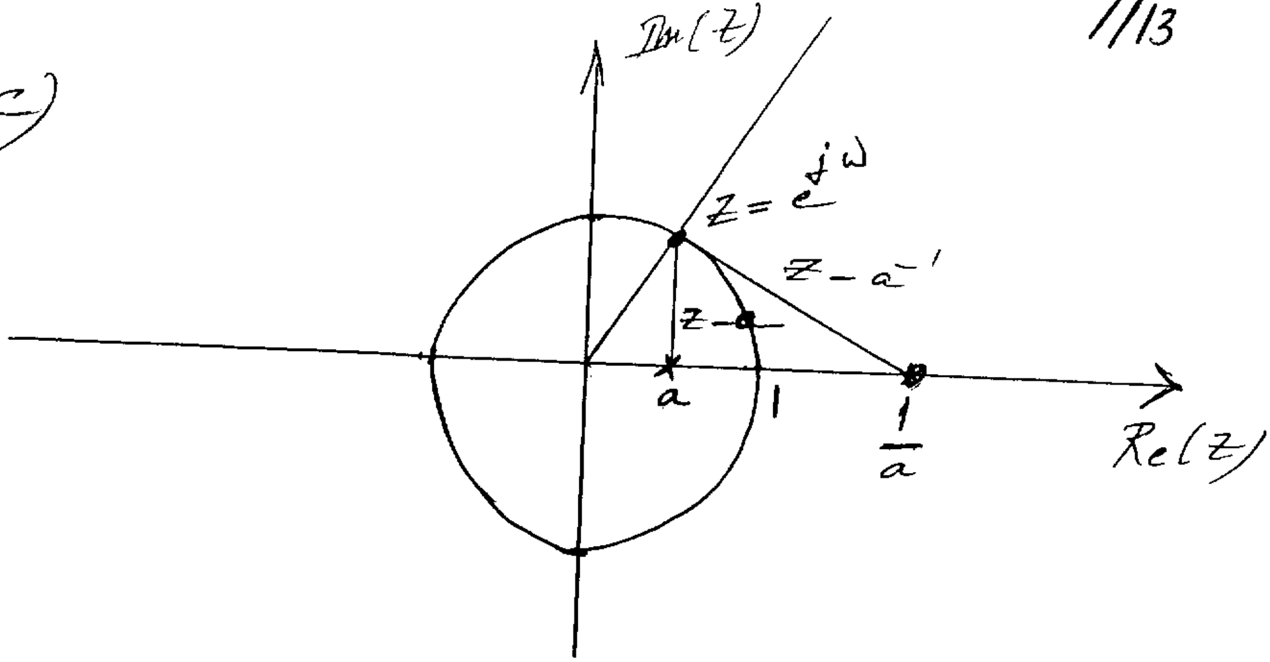
3) $H(z) = \dfrac{1 - a^{-1} z^{-1}}{1 - a z^{-1}} = \dfrac{z - a^{-1}}{z - a}, \quad |z| > |a|$

a) the system is stable for $|a| < 1 \Rightarrow -1 < a < +1$

b)



c)



$$|z-a^{-1}|^2 = 1+\left(\frac{1}{a}\right)^2 - 2\left(\frac{1}{a}\right)\cos\omega$$

$$= \frac{a^2+1-2a\cos\omega}{a^2}$$

$$= \frac{1}{a^2}|z-a|^2$$

$$\Rightarrow |H(\omega)| = \frac{|z-a^{-1}|}{|z-a|} = \left|\frac{1}{a}\right| = \text{constant.}$$

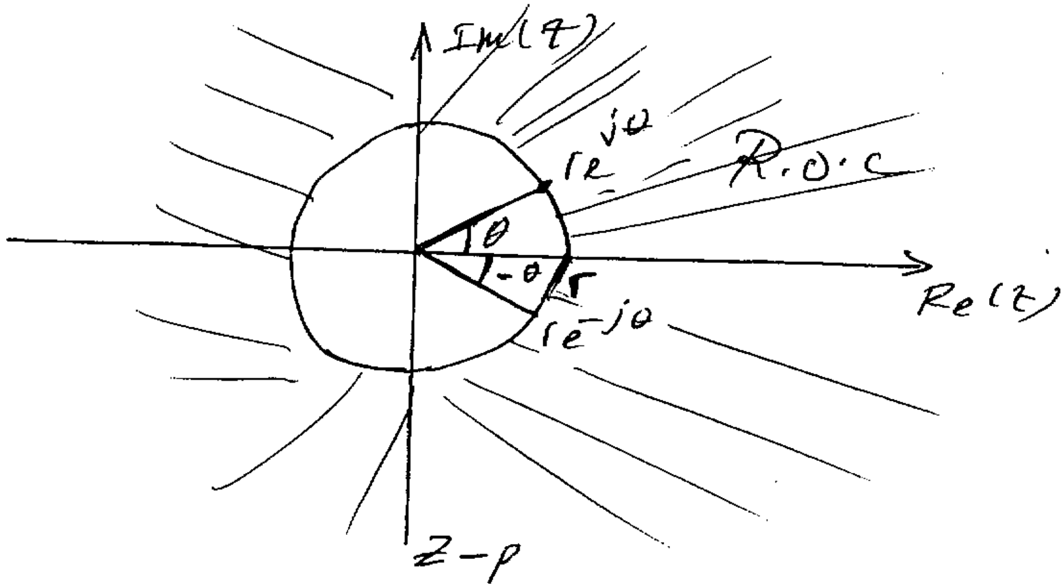
4) $y(n) - 2r\cos\theta\, y(n-1) + r^2 y(n-2) = x(n)$.

$$Y(z)\left[1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$

$$= \frac{z^2}{z^2 - 2r\cos\theta\, z + r^2}$$

$$= \frac{z^2}{(z - re^{j\theta})(z - re^{-j\theta})}, \quad |z| > |r|$$



$x(n) = \alpha^n u(n) \Rightarrow$

$$X(z) = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}, \quad |z| > |\alpha|$$

$$Y(z) = H(z)X(z)$$

$$= \frac{z^3}{(z-\alpha)(z-re^{j\theta})(z-re^{-j\theta})}, |z| > \max(|r|, |\alpha|)$$

$$y(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n+2}}{(z-\alpha)(z-re^{j\theta})(z-re^{-j\theta})}dz$$

$$= \left.\frac{z^{n+2}}{(z-\alpha)(z-re^{j\theta})}\right|_{z=re^{-j\theta}} + \left.\frac{z^{n+2}}{(z-\alpha)(z-re^{-j\theta})}\right|_{z=re^{j\theta}}$$

$$+ \left.\frac{z^{n+2}}{(z-re^{j\theta})(z-re^{-j\theta})}\right|_{z=\alpha}$$

$$= \frac{r^{n+2}e^{-j(n+2)\theta}}{(re^{-j\theta}-\alpha)(re^{-j\theta}-re^{j\theta})}$$

$$+ \frac{r^{n+2}e^{j(n+2)\theta}}{(re^{j\theta}-\alpha)(re^{j\theta}-re^{-j\theta})}$$

$$+ \frac{\alpha^{n+2}}{(\alpha-re^{j\theta})(\alpha-re^{-j\theta})}$$

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$$= \frac{(\alpha - re^{j\theta})r^{n+2}e^{-j(n+2)\theta} + (re^{-j\theta}-\alpha)r^{n+2}e^{j(n+2)\theta} + 2jr\sin\theta\,\alpha^{n+2}}{(re^{-j\theta}-\alpha)(re^{j\theta}-re^{-j\theta})(re^{j\theta}-\alpha)}$$

$$= \frac{r^{n+2}\left[(\alpha - re^{j\theta})e^{-j(n+2)\theta} + (re^{-j\theta}-\alpha)e^{j(n+2)\theta}\right] - 2jr\sin\theta\,\alpha^{n+2}}{(re^{-j\theta}-\alpha)(re^{j\theta}-\alpha)(re^{j\theta}-re^{-j\theta})}$$

$n \geq 0$

$y(n) = 0, \quad n < 0.$

5) $H(z) = \dfrac{1 - z^{-1}}{1 - 2z^{-1} + .75z^{-2}}$

$$= \frac{z(z-1)}{z^2 - 2z + .75}$$

$$= \frac{z(z-1)}{(z-.5)(z-1.5)}$$

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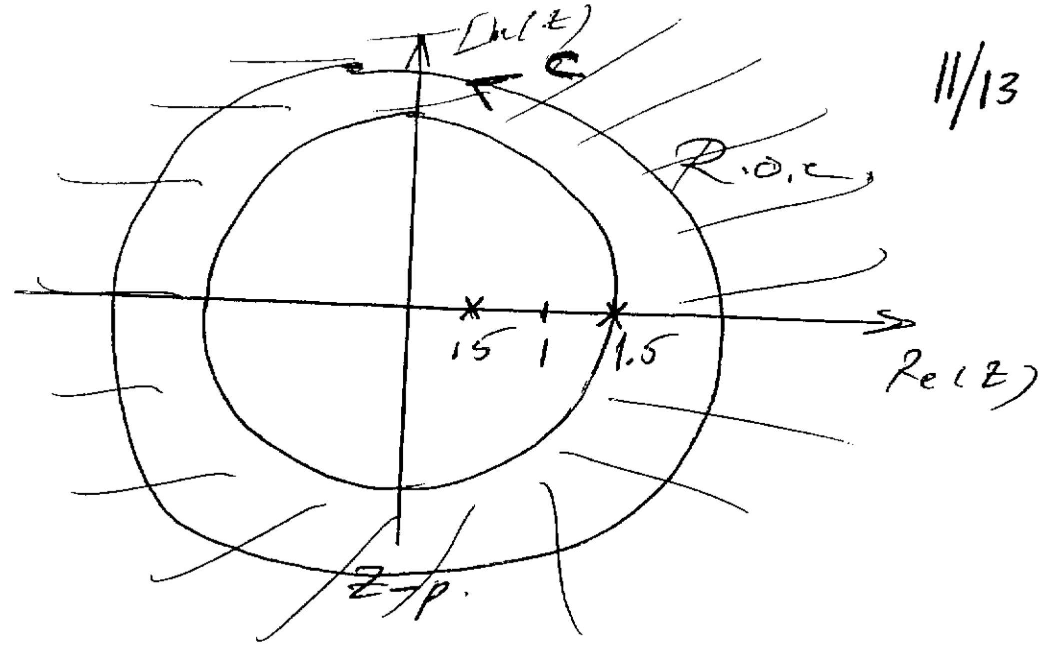
a)



$$h(n) = \frac{1}{2\pi j}\oint_c \frac{(z-1)z^n}{(z-.5)(z-1.5)}dz$$

$$= \left.\frac{(z-1)z^n}{(z-.5)}\right|_{z=1.5} + \left.\frac{(z-1)z^n}{(z-1.5)}\right|_{z=.5}$$

$$= \left[.5(1.5)^n + (.5)^{n+1}\right]u(n).$$

Or. using partial fractions:

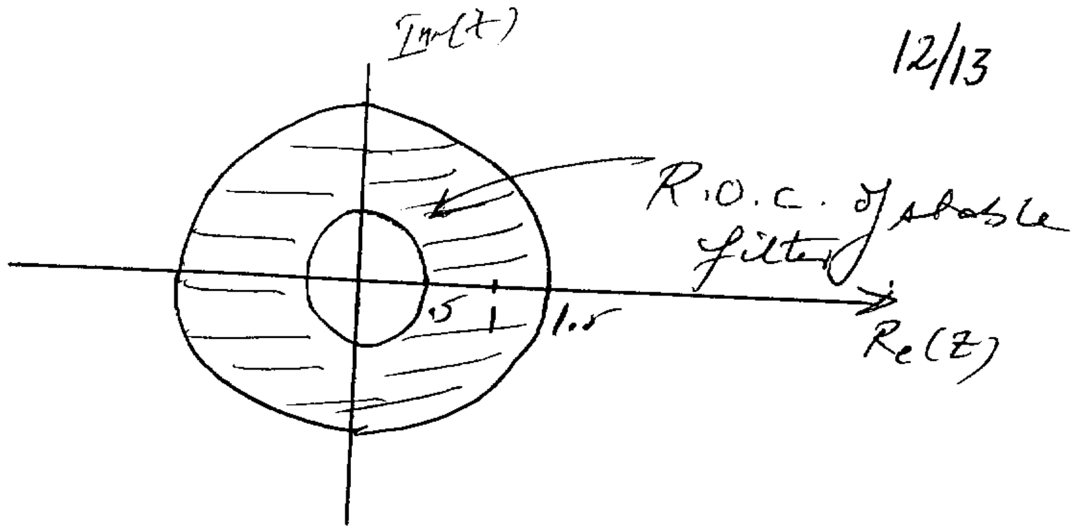
$$H(z) = 1 + \frac{.25}{(z-.5)} + \frac{.75}{(z-1.5)}$$

$$= 1 + \frac{.25z^{-1}}{1-.5z^{-1}} + \frac{.75z^{-1}}{1-1.5z^{-1}}$$

$$h(n) = \delta(n) + (.5)^{n+1}u(n-1) + .5(1.5)^n u(n-1)$$

$$= \left[.5(1.5)^n + (.5)^{n+1}\right]u(n).$$

b)



z-p

$$h(n) = \frac{1}{2\pi j}\oint_C \frac{(z-1)z^n}{(z-.5)(z-1.5)}\,dz$$

$$= \left.\frac{(z-1)z^n}{(z-1.5)}\right|_{z=.5} = (.5)^{n+1}u(n).$$

$$H(p^{-1}) = \frac{1-p}{.75p^2-2p+1}$$

$$= \frac{1-p}{.75(p-2)(p-\frac{1}{1.5})}, \quad \frac{1}{1.5}<|p|<2$$

$$h(n) = \frac{1}{2\pi j}\oint_C \frac{(1-p)p^{-n-1}}{(.75)(p-2)(p-\frac{1}{1.5})}\,dp$$

$$= \frac{(1-p)\,p^{-n-1}}{.75\,(p-2)}\Bigg|_{p=\frac{1}{1.6}} = -.5\,(1.5)^{n}\,u(-n-1)$$