

Problem #1

$$x(n) = 2 \cos\left(\frac{11\pi}{3}n + \frac{\pi}{4}\right)$$

For this sinusoidal signal to be periodic, an integer k needs to exist such that $k(2\pi/\omega_0)$ is an integer. Hence, $k \times 2\pi/\frac{11\pi}{3} = \frac{6k}{11}$ is an integer when $k = 11, 22, 33, \dots$ and the signal is periodic with period obtained using the smallest k . Thus, $n_p = 6$.

Problem #2

$$x(n) = 2 \cos\left(\frac{11\pi}{3}n + \frac{\pi}{4}\right)$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

If $x(n)$ is present at the input of an LTI discrete-time system with impulse response $h(n)$, then the steady-state output $y_s(n)$ becomes:

$$y_s(n) = 2 |H(\omega_0)| \cos\left(\frac{11\pi}{3}n + \frac{\pi}{4} + \arg H(\omega_0)\right)$$

where $\omega_0 = 11\pi/3$, and $H(\omega)$ is the system frequency response.

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n \\ &= \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \end{aligned}$$

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$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{1}{1 - \frac{1}{2}\cos(\omega) + \frac{1}{2}j\sin(\omega)}$$

$$H\left(\frac{11\pi}{3}\right) = \frac{1}{1 - \frac{1}{4} - j\frac{\sqrt{3}}{4}} = \frac{1}{\frac{3}{4} - j\frac{\sqrt{3}}{4}}$$

$$|H\left(\frac{11\pi}{3}\right)| = \left[\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2\right]^{-\frac{1}{2}} = \left(\frac{16}{16}\right)^{-\frac{1}{2}} = \frac{2}{\sqrt{3}}$$

$$\begin{aligned} \arg\left(H\left(\frac{11\pi}{3}\right)\right) &= 0 - \tan^{-1}\left(-\frac{\sqrt{3}}{4} \times \frac{4}{3}\right) = -\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) \\ &= \frac{\pi}{6}. \end{aligned}$$

$$\begin{aligned} \Rightarrow y(n) &= \frac{4}{\sqrt{3}} \cos\left(\frac{11\pi}{3}n + \frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \frac{4}{\sqrt{3}} \cos\left(\frac{11\pi}{3}n + \frac{5\pi}{12}\right) \end{aligned}$$

Problem # 3 A linear and time-varying discrete-time system is given by:

$$y(n) = T[x(n)] = a^n x(n)$$

$x(n) = b^n u(n)$ is the system input.

The system output becomes:

$$y(n) = a^n b^n u(n) = (ab)^n u(n).$$

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Problem #4

$$y(n) = T[x(n)] = a^n x(n)$$

$$h(n) = T[\delta(n)] = a^n \delta(n) = \delta(n)$$

Since $\delta(n) = \begin{cases} 1, & n=0 \\ 0, & \text{elsewhere.} \end{cases}$

$$\begin{aligned} y_c(n) &= x(n) * h(n) = x(n) * \delta(n) = x(n) \\ &= b^n u(n) \end{aligned}$$

$\neq (ab)^n u(n)$, which is the system output.

Of course, convolution does not give the actual system output since the system is linear but not time-invariant.

Problem #5

$$H(s) = \frac{s-1}{(s+2)(s+5)}$$

$$H(s) = \frac{A}{s+2} + \frac{B}{s+5}; A = \left. \frac{s-1}{(s+5)} \right|_{s=-2} = -1$$

$$B = \left. \frac{s-1}{s+2} \right|_{s=-5} = 2$$

$$\Rightarrow h(t) = \left[e^{-2t} + 2e^{-5t} \right] u(t)$$

$$h(n) = h(t) \Big|_{t=n} = \left[e^{2n} + 2e^{-5n} \right] u(n).$$

$$= -a^n u(n) + 2b^n u(n), \text{ with } a = e^{-2} \\ b = e^{-5}$$

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$$\begin{aligned}
 H(z) &= -\sum_{n=0}^{\infty} (az^{-1})^n + z \sum_{n=0}^{\infty} (bz^{-1})^n \\
 &= -\frac{1}{1-az^{-1}} + \frac{z}{1-bz^{-1}}, |z| > \max[|a|, |b|] \\
 &= -\frac{z}{z-a} + \frac{az}{z-b} \\
 &= \frac{-z(z-b) + az(z-a)}{(z-a)(z-b)} \\
 &= \frac{z^2 - (2a-b)z}{(z-a)(z-b)}, |z| > |a|
 \end{aligned}$$

Note $|a| = e^{-\alpha} < 1$. Hence, the ROC of $H(z)$ contains the unit circle.

Also, the poles of $H(z)$; i.e., $z=a$ and $z=b$ are inside the unit circle.

Hence, the discrete-time system is causal and stable.

Problem # 6

$$x(n) = a^{n-1} u(n-1) + b^{n-2} u(n-2)$$

Here, $X(z)$ can be determined using its direct formula or using the linearity and time shift properties of the Z-transform.

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$$\begin{aligned}
 X(z) &= z^{-1}Z(a^n u(n)) + z^{-2}Z(b^n u(n)) \\
 &= z^{-1} \times \frac{z}{z-a} + z^{-2} \times \frac{z}{z-b} \\
 &= \frac{z-b+z^{-1}(z-a)}{(z-a)(z-b)} \\
 &= \frac{z-b+1-\alpha z^{-1}}{(z-a)(z-b)} \\
 &= \frac{z^2 - z(b-1) - \alpha}{z(z-a)(z-b)}, |z| > \max(|\alpha|, |b|)
 \end{aligned}$$

Problem #7

$$\begin{aligned}
 x(n) &= a^{n-1} u(n-1) \\
 h(n) &= a^n u(n)
 \end{aligned}$$

$$Y(z) = H(z)X(z)$$

$$X(z) = z^{-1} \times \frac{z}{z-a} = \frac{1}{z-a}, |z| > |\alpha|$$

$$H(z) = \frac{z}{z-a}, |z| > |\alpha|$$

$$\Rightarrow Y(z) = \frac{z}{(z-a)^2}, |z| > |\alpha|$$

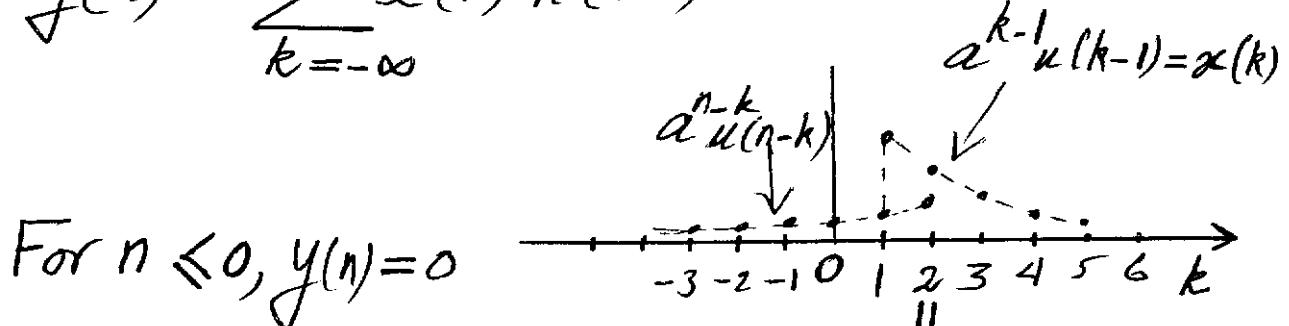
$$\begin{aligned}
 y(n) &= \frac{1}{2\pi j} \oint_C \frac{z \cdot z^{n-1}}{(z-a)^2} dz \\
 &= \left. \frac{d}{dz} (z^n) \right]_{z=a} = n z^{n-1} \Big|_{z=a} = n a^{n-1}, n \geq 0
 \end{aligned}$$

$$y(n) = n a^{n-1} u(n) = n a^{n-1} u(n-1)$$

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Using convolution we have:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$



$$\text{For } n \leq 0, y(n) = 0$$

$$\begin{aligned} \text{For } n \geq 1, y(n) &= \sum_{k=1}^n a^{k-1} a^{n-k} \\ &= \sum_{k=1}^n a^{n-1} = na^{n-1} \end{aligned}$$

$$\text{Hence, } y(n) = na^{n-1} u(n-1).$$

Problem # 8

$$h(n) = (a^n + b^n) u(n)$$

$$\begin{aligned} \sum_{n=0}^{\infty} |h(n)| &= \sum_{n=0}^{\infty} |a^n + b^n| \leq \sum_{n=0}^{\infty} [|a^n| + |b^n|] \\ &\leq \sum_{n=0}^{\infty} |a|^n + \sum_{n=0}^{\infty} |b|^n < \infty \end{aligned}$$

if $|a| < 1$ and $|b| < 1$ or $\max(|a|, |b|) < 1$.
Using the frequency domain analysis:

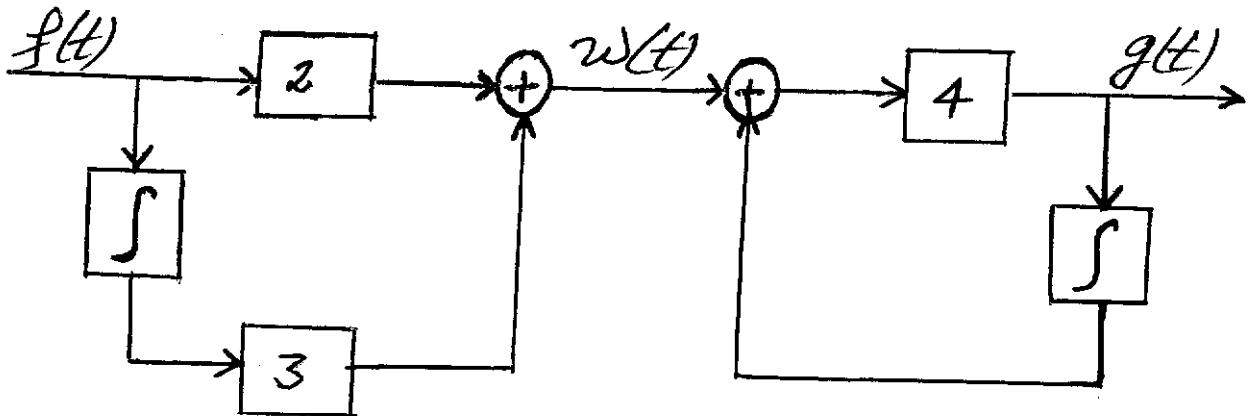
$$H(z) = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (bz^{-1})^n = \frac{z}{z-a} + \frac{z}{z-b}$$

with $|z| > |a|$ and $|z| > |b| \Leftrightarrow |z| > \max(|a|, |b|)$.

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The region of convergence of $H(z)$ contains the unit circle if $\max(|a_1|, |b_1|) < 1$. This condition makes the system stable. It also leads to having the poles of $H(z)$ inside the unit circle. Note here that the system is causal.

Problem # 9



$$w(t) = 2f(t) + 3 \int f(t) dt$$

$$\begin{aligned} g(t) &= 4[w(t) + \int g(t) dt] \\ &= 4[2f(t) + 3 \int f(t) dt + \int g(t) dt] \end{aligned}$$

$$\Rightarrow g(t) - 4 \int g(t) dt = 8f(t) + 12 \int f(t) dt$$

$$\Rightarrow \frac{dg(t)}{dt} - 4g(t) = 8 \frac{df(t)}{dt} + 12f(t)$$

Problem # 10

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1) Using the system differential equation and the Laplace transform applied to both sides of this equation, we obtain:

$$SG(s) - 4G(s) = 8sF(s) + 12F(s)$$

$$G(s)[s-4] = F(s)[8s+12]$$

$$\Rightarrow H(s) = \frac{G(s)}{F(s)} = \frac{8s+12}{s-4}$$

2) Using the block diagram, we can write:

$$W(s) = 2F(s) + 3\frac{F(s)}{s}$$

$$G(s) = 4\left[W(s) + \frac{G(s)}{s}\right]$$

$$\Rightarrow G(s)\left[1 - \frac{4}{s}\right] = 4W(s) = 8F(s) + 12\frac{F(s)}{s}$$

$$\text{or } G(s)\left[\frac{s-4}{s}\right] = F(s)\left[\frac{8s+12}{s}\right] = F(s)\left[8 + \frac{12}{s}\right]$$

$$H(s) = \frac{G(s)}{F(s)} = \frac{8s+12}{s-4}$$

The system transfer function has a pole at $s=4$, i.e., in the right half s-plane. Hence, the system is unstable since it is causal and implemented by the block diagram shown in Problem # 9.