EECE 440 – SPRING 2007
QUIZ 2 SOLUTIONS

KARA

PROBLEM 1

$$y(n) = x(n) - \frac{3}{2}x(n-1) - x(n-2)$$

a) This system has delayed versions of the input represented in the output, such as echo, sound reflections etc.

4 b) system is an FIR causal DT LTI system

FIR $\Rightarrow$ $\sum |h(n)| < \infty$ $\Rightarrow$ stable.

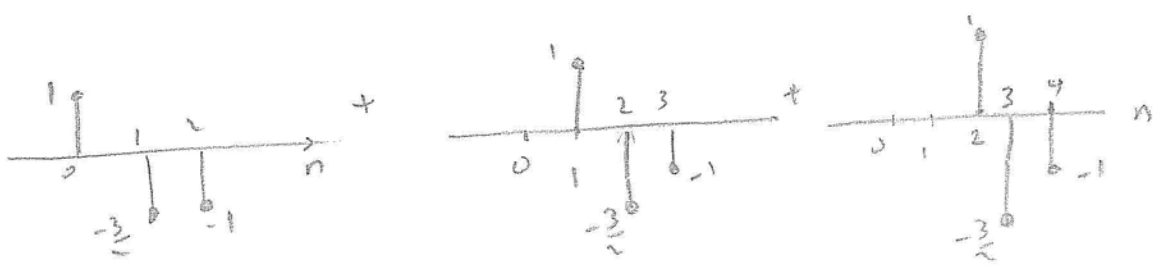
it is also causal $\quad (y(n) = f\{x(n-i)\} \quad i \geq 0)$

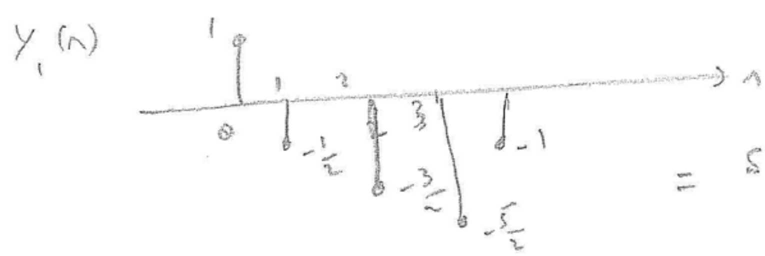
3 c) $h(n) = \delta(n) - \frac{3}{2}\delta(n-1) - \delta(n-2)$

3 d) $x_1(n) = u(n) - u(n-3) \longrightarrow y_1(n) = ?$

$\Rightarrow y_1(n) = h(n) + h(n-1) + h(n-2)$

by LTI





$$= \delta(n) - \frac{1}{2}\delta(n-1) - \frac{3}{2}\delta(n-2) - \frac{5}{2}\delta(n-3) - \delta(n-4)$$

5 e) $$\frac{Y_{new}(z)}{X(z)} = \frac{K\,H(z)}{1 + K\,H(z)} \Rightarrow \frac{\frac{a}{b + cz^{-1} + dz^{-2}} \cdot \left(1 - \frac{3}{2}z^{-1} - z^{-2}\right)}{1 + \frac{a}{b + cz^{-1} + dz^{-2}}\left(1 - \frac{3}{2}z^{-1} - z^{-2}\right)}$$

PROBLEM 2

a) 1- $y[n] - \frac{1}{4} y[n-1] = x[n]$ $\qquad x(n) = \left(\frac{1}{2}\right)^n u(n)$

$$\left(1 - \frac{1}{4} z^{-1}\right) Y(z) = X(z) \Rightarrow Y(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} \cdot \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$$

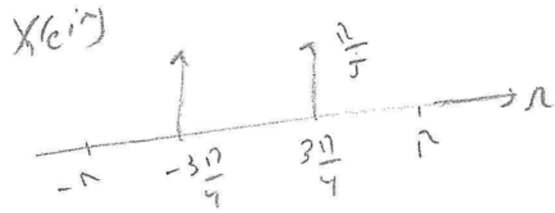
$$Y(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}} \qquad A = \frac{1}{1-\frac{2}{4}} = 2$$

$$B = \frac{1}{1-\frac{4}{2}} = -1$$

$$\Rightarrow Y(z) = \frac{2}{1-\frac{1}{2}z^{-1}} \; \frac{-\frac{1}{2}}{1-\frac{1}{4}z^{-1}}$$

$$\Rightarrow y_{ZSR}(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n)$$

2- $x[n] = \sin \frac{3\Omega}{4} n \Rightarrow X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \frac{\pi}{j}\left[\delta\left(\Omega - \frac{3\Omega}{4} - 2k\pi\right) - \delta\left(\Omega + \frac{3\Omega}{4} + 2\pi k\right)\right]$



Since system is stable

$$H(e^{j\Omega}) = H(z)\Big|_{z=e^{j\Omega}}$$

$$= \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = H(e^{j\Omega}) \cdot X(e^{j\Omega}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left[\frac{1}{1-\frac{1}{4}e^{-j\left(\frac{3\Omega}{4}-2\pi k\right)}} \delta\left(\Omega - \frac{3\Omega}{4} - 2\pi\right.\right.$$

$$- \frac{1}{1-\frac{1}{4}e^{-j\left(\frac{3\Omega}{4}-2\pi k\right)}} \delta\left(\Omega + \frac{3\Omega}{4}\right.$$

(1)

b) 1. $y(n) = u(n) * \left(\frac{1}{2}\right)^n u(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} u(n-k) \left(\frac{1}{2}\right)^k u(k) = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^k u(k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$= 2\left(1-\left(\frac{1}{2}\right)^{n+1}\right) u(n)$$

2. $$z(t) = \int_{-\infty}^{\infty} (\lambda^2 + 8\lambda + \cos\lambda)\, u(t-\lambda)\, d\lambda = \int_{-\infty}^{t} (\lambda^2 + 8\lambda + \cos\lambda)\, d\lambda$$

$$= \underbrace{\frac{t^3}{3} + 4t^2 + \sin\lambda \Big|_{\lambda=-\infty}^{t}}$$

$$\cos t * u(t) \longleftrightarrow \pi\left[\delta(\omega-1) + \delta(\omega+1)\right] \cdot \left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]$$

$$\sin t \longleftrightarrow \frac{\pi}{j}\left[\delta(\omega-1) - \delta(\omega+1)\right] + \ldots$$

$$= \frac{t^3}{3} + 4t^2 + \sin t$$

c) System is CAUSAL + STABLE $\Rightarrow G(j\omega) = G(s)\big|_{s=j\omega} = G(\omega)$

$$\Rightarrow G(s) = \frac{s+4}{6+s^2+5s} = \frac{s+4}{(s+3)(s+2)} = \frac{Y(s)}{X(s)}$$

$$\Rightarrow \left[\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y(t) = \frac{dx}{dt} + 4x(t)\right]$$

$g(t) = ?$ $\quad G(s) = \frac{A}{s+3} + \frac{B}{s+2}$ $\quad A = \frac{-3+4}{-3+2} = -1$

$B = \frac{-2+4}{-2+3} = 2$

$$\Rightarrow G(s) = \frac{-1}{s+3} + \frac{2}{s+2}$$

$$\Rightarrow g(t) = \left(2e^{-2t} - e^{-3t}\right) u(t)$$

PROBLEM 3

$$y(n) - \frac{3}{2} y(n-1) - y(n-2) = x(n) + 2x(n-1)$$

1) $$y(n) - \frac{3}{2} y(n-1) - y(n-2) = w(n) \quad \text{system 1}$$

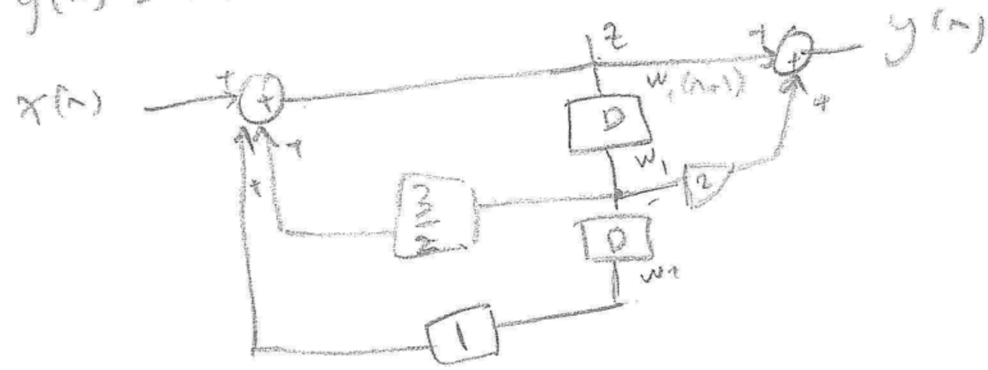
$$w(n) = x(n) + 2x(n-1) \quad \text{system 2}$$

flip order

$\Rightarrow$

$$z(n) - \frac{3}{2} z(n-1) - z(n-2) = x(n) \quad \rightarrow \quad z(n) = x(n) + \frac{3}{2} z(n-1) + z(n-2)$$

and $$y(n) = z(n) + 2z(n-1)$$



2) $$w_1(n+1) = \frac{3}{2} w_1(n) + w_2(n) + x(n)$$

$$w_2(n+1) = w_1(n)$$

$$y[n] = \frac{3}{2} w_1(n) + w_2(n) + x(n) + 2w_1(n)$$

$$\begin{bmatrix} w_1(n+1) \\ w_2(n+1) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1(n) \\ w_2(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} \frac{7}{2} & 1 \end{bmatrix} \begin{bmatrix} w_1(n) \\ \dots \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(n)$$

a) Take Z XFORM

$$\left(1 - \frac{3}{2}z^{-1} - z^{-2}\right) Y(z) = \left(1 + 2z^{-1}\right) X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+2z^{-1}}{1-\frac{3}{2}z^{-1}-z^{-2}} = \frac{1+2z^{-1}}{\left(1-2z^{-1}\right)\left(1+\frac{1}{2}\right.}$$

$$H(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} \qquad A = \frac{1+2\left(\frac{1}{2}\right)}{1+\frac{1}{4}} = \frac{8}{5}$$

$$B = \frac{1+2(-2)}{1-2(-2)} = \frac{-3}{5}$$

$$\Rightarrow h(n) = -\frac{8}{5}(2)^n u(-n-1) - \frac{3}{5}\left(-\frac{1}{2}\right)^n u(n)$$

R.O.C $\quad \frac{1}{2} < |z| < 2$

b) $$x(n) \longrightarrow \boxed{h_1(n)} \xrightarrow{z(n)} \boxed{h_2(n)} \longrightarrow y(n)$$

$$x(n) = \delta(n) - \frac{1}{3}\delta(n-1) \qquad \begin{aligned} h_1(n) &= \sin(20n) \\ h_2(n) &= \left(\frac{1}{3}\right)^n u(n) \end{aligned}$$

flip order !!

$$x \longrightarrow \boxed{h_2} \xrightarrow{w} \boxed{h_1} \longrightarrow y$$

$$w(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

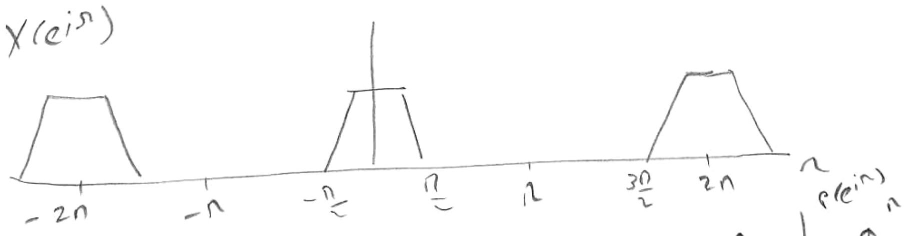
$$= \left(\frac{1}{3}\right)^n \left[u(n) - u(n-1)\right] = \left(\frac{1}{3}\right)^0 \delta(n) = \delta(n)!$$

$$\Rightarrow \quad y(n) = \sin(20n)$$

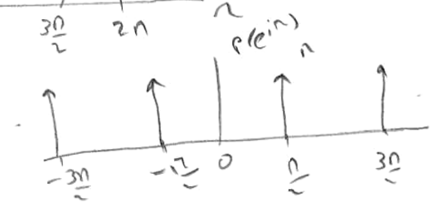
(5)

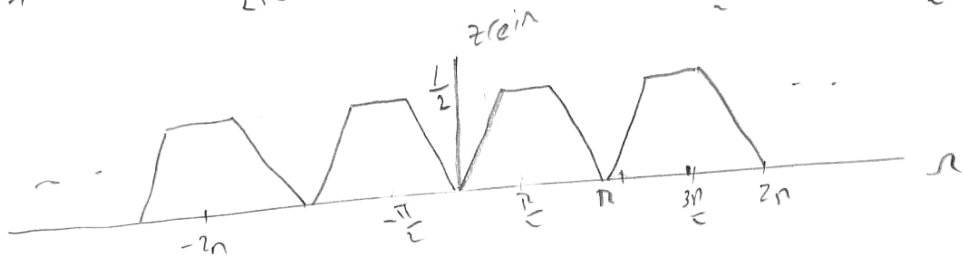
c) $z[n] = x[n] \cdot p[n]$

$p[n] = \cos\frac{\pi n}{2} \rightarrow P(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{2} - 2\pi k) + \delta(\Omega + \frac{\pi}{2} - 2\pi k)$



$\Rightarrow Z(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast P(e^{j\Omega})$





PROBLEM 4

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$h(t) = e^{-4|t|}$

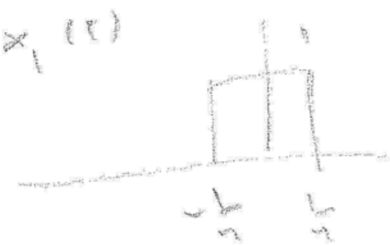
$= e^{4t}u(-t) + e^{-4t}u(t) \Rightarrow H(s) = \frac{-1}{s-4} + \frac{1}{s+4}$

$\Rightarrow H(s) = \frac{-s-4+s-4}{s^2-16} = \frac{-8}{s^2-16}$

system stable $\Rightarrow H(\omega) = H(s)\big|_{s=j\omega} = \frac{8}{\omega^2+16}$

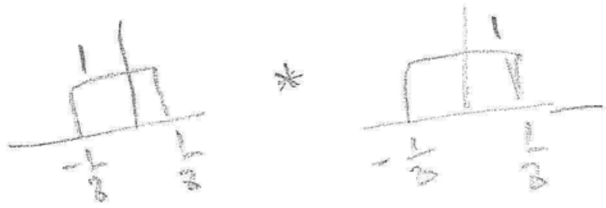
$x(t)$ is periodic. Take one period

$x_T(t) = x_1(t) + x_2(t)$



$X_1(\omega) = \frac{1}{2}\frac{\sin\frac{\omega}{4}}{\frac{\omega}{4}} = \frac{1}{2}\,\mathrm{sinc}\left(\frac{\omega}{4}\right)$

$X_2(t) =$



$\times 2$

$\Rightarrow X_2(\omega) = 2\left(\frac{1}{4}\frac{\sin\frac{\omega}{8}}{\frac{\omega}{8}}\right)^2 = \frac{1}{8}\,\mathrm{sinc}^2\left(\frac{\omega}{8}\right)$

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or $X(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} X_T\left(\frac{2\pi}{T}k\right) \delta\left(\omega - \frac{2\pi}{T}k\right)$ $T=1$

$$= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\frac{\sin\frac{\pi k}{2}}{\frac{\pi k}{2}} + \frac{1}{8}\frac{\sin^2\frac{\pi k}{4}}{\frac{\pi^2 k^2}{4}} \right) \delta(\omega - 2\pi k)$$

$Y(\omega) = H(\omega) \cdot X(\omega)$

$$= \frac{8}{\omega^2+16} \cdot \sum_{k=-\infty}^{\infty} \left(\frac{2\sin\frac{\pi k}{2}}{k} + \frac{4\sin^2\frac{\pi k}{4}}{\pi k^2} \right) \delta(\omega - 2\pi k)$$

$g(t) = \frac{\sin\frac{\pi}{16}t}{\pi t} \longrightarrow G(\omega)$

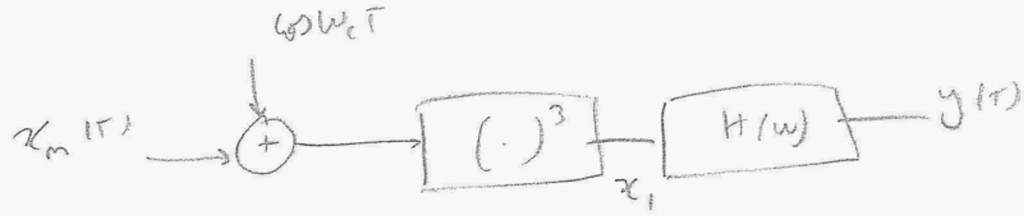
$Z(\omega) = G(\omega) \cdot Y(\omega)$

$Y(\omega)$ has $\delta(\Omega - 2\pi k)$ $k = 0, \pm 1, \pm 2$ ... only $k=0$ survives $G(\omega)$

$\Rightarrow Z(\omega) = \frac{8}{16}(2\pi)\left(\frac{1}{2}+\frac{1}{8}\right) \cdot \delta(\omega) = 2\pi \cdot \frac{5}{16} \cdot \delta(\omega)$

$\Rightarrow \left[z(t) = \frac{5}{16} \right]$

PROBLEM 5

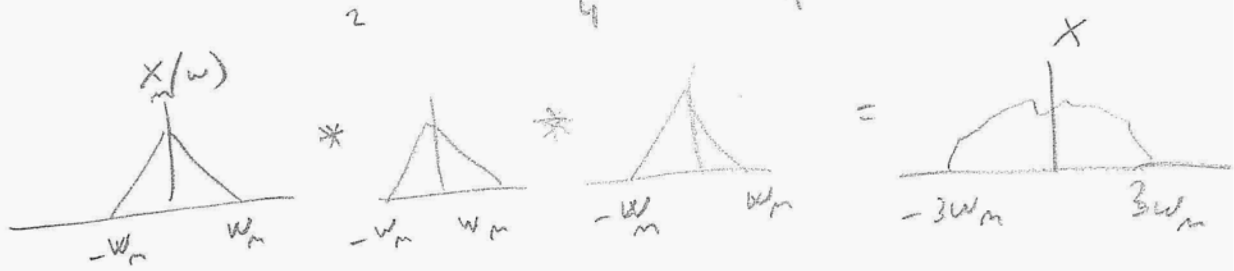


$$x_1(t) = (x_m + \cos w_c t)^3 = x_m^3(t) + 3x_m^2 \cos w_c t + 3x_m \cos^2 w_c t + \cos^3 w_c t$$

$$x_1(t) = x_m^3(t) + 3x_m^2(t)\cos w_c t + 3x_m(t)\left(\frac{1+\cos 2w_c t}{2}\right) + \frac{(1+\cos 2w_c t)}{2}\cdot \cos w_c t$$

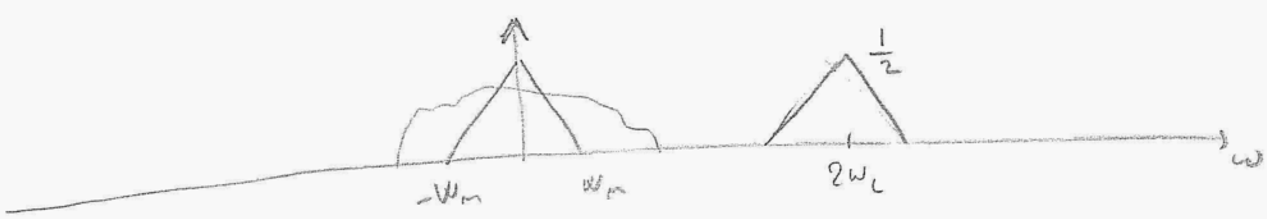
$$x_1(t) = x_m^3(t) + \frac{3}{2}x_m(t) + \frac{3}{2}x_m(t)\cos 2w_c t + 3x_m^2(t)\cos w_c t + \frac{\cos w_c t}{2} + \frac{\cos 3w_c t}{4} + \frac{\cos w_c t}{4}$$

$x_m^3(t)$

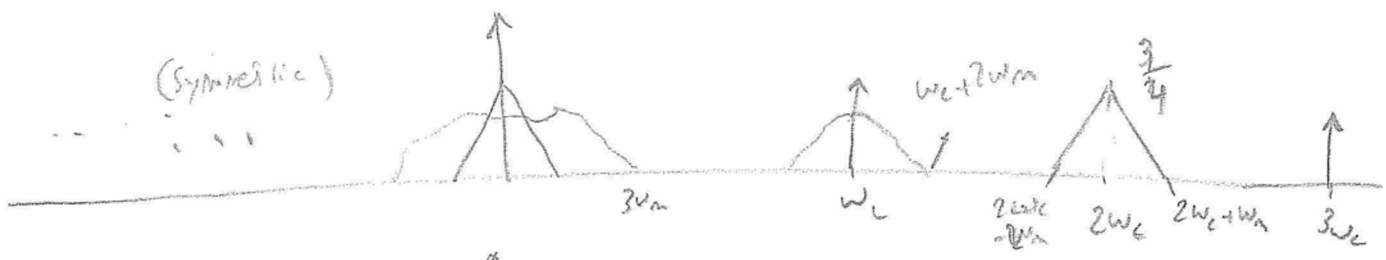


$x_m^2(t)$

So the Fourier xform looks like



$$x_1(t) = \underbrace{x_m^3(t) + \frac{3}{2}x_m(t)}_{\text{at dc}} + \underbrace{\frac{3}{4}\cos w_c t + 3x_m^2(t)\cos w_c t}_{\text{at } w_c}$$



$x_m^3(t) + \frac{3}{2}x_m(t)$

$3x_m^2(t)\cos w_c t + \frac{3}{4}\cos w_c t$

$\frac{3}{2}x_m(t)\cos 2w_c t$

$\frac{1}{4}\cos 3w_c t$

To work

$$\left.\begin{array}{ll} w_c+2w_m < 2w_c - w_m \Rightarrow & 3w_m < w_c \\ 2w_c + w_m < 3w_c \Rightarrow & w_m < w_c \end{array}\right\} \Rightarrow \boxed{w_c > 3w_m}$$

if we choose $w_L = 2w_c - w_m$, $A = \frac{2}{3} \Rightarrow$ obtain a modulated form of $x_m(t)$

In general, need w_L to be in the range:

$(w_c + 2w_m < w_L)$ ① and $(w_L < 2w_c - w_m)$ ② and

$(2w_c + w_m < w_H < 3w_c)$ ③ $\quad (w_H = 2w_c + 2w_c - w_L = 4w_c - w_L)$

③ $\Rightarrow 2w_c + w_m < 4w_c - w_L < 3w_c \Rightarrow \boxed{w_L > w_c}$

② $\Rightarrow \boxed{w_L < 2w_c - w_m}$

① $\Rightarrow \boxed{w_L > w_c + 2w_m}$ Note: ③ is included in ①

$$\Rightarrow \left[w_c + 2w_m < w_L < 2w_c - w_m \right]$$

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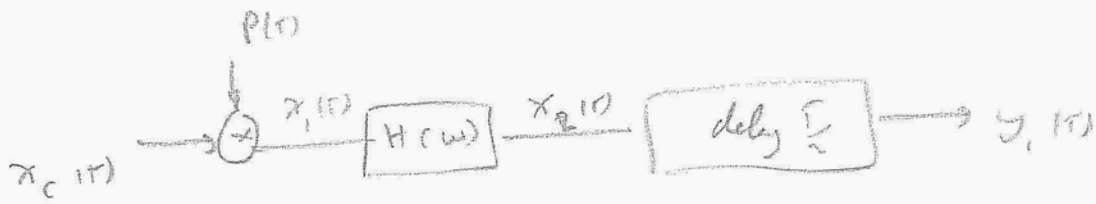
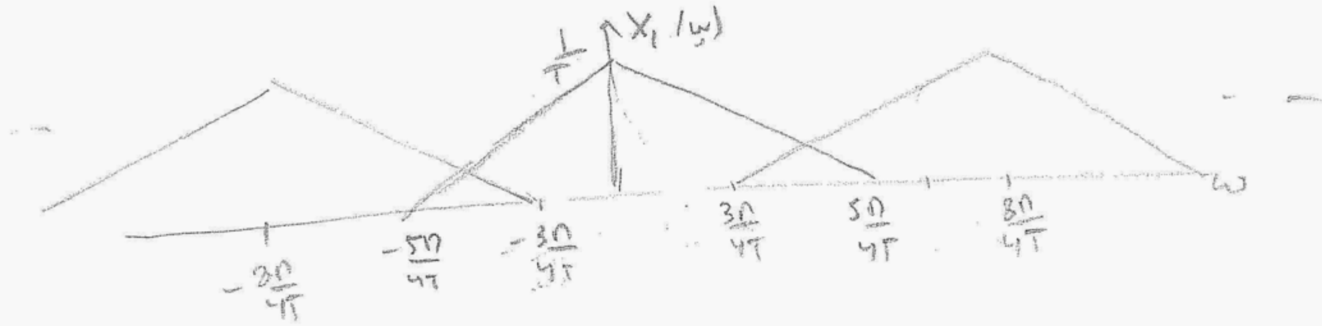
problem 6)

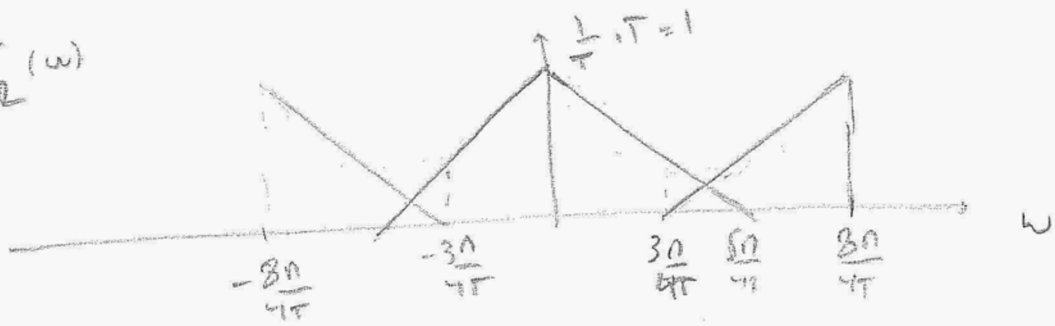


a) Nyquist $2(W_{max}) = \frac{5\pi}{2T}$ rad/sec

b) $x_1(t) = x_c(t)\cdot p(t) \Rightarrow X_1(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\omega - \frac{2\pi}{T}k\right)$



$\Rightarrow X_2(\omega)$



$Y_2(\omega) = X_2(\omega)\cdot e^{-j\omega\frac{T}{2}}$

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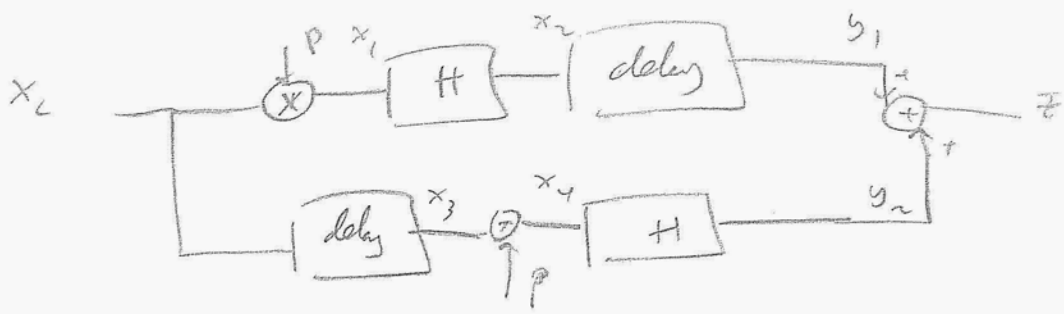


$x_3(t) = x_c(t - \frac{T}{2})$

$\Rightarrow X_3(\omega) = e^{-j\omega\frac{T}{2}} \cdot X_c(\omega)$

$x_4(t) = x_3(t) \cdot p(t) \Rightarrow X_4(\omega) = \frac{1}{T} \sum_{-\infty}^{\infty} X_3(\omega - \frac{2\pi}{T}k)$

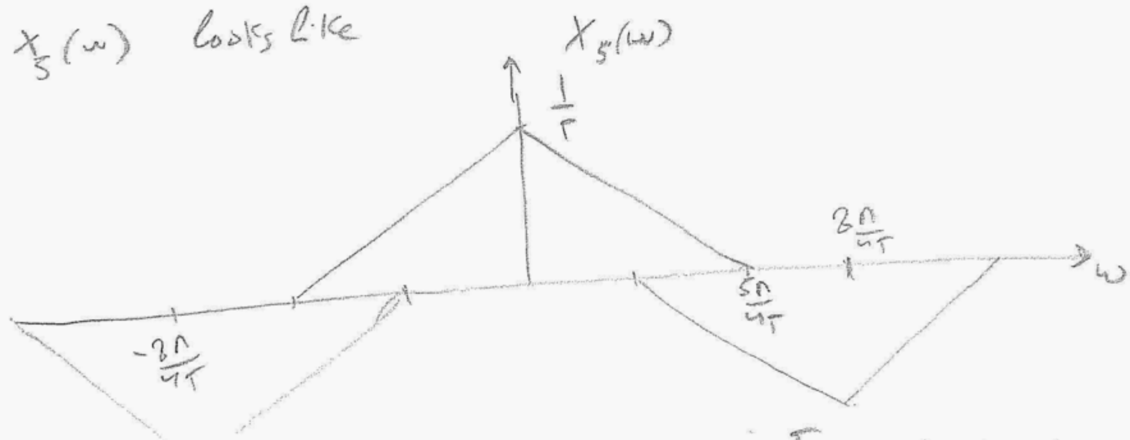
$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\omega - \frac{2\pi}{T}k) \cdot e^{-j(\omega - \frac{2\pi}{T}k)\frac{T}{2}}$

$= \frac{1}{T} e^{-j\omega\frac{T}{2}} \sum_{k=-\infty}^{\infty} e^{+jk\pi} \cdot X_c(\omega - \frac{2\pi}{T}k)$

$\Rightarrow X_4(\omega) = \frac{e^{-j\omega\frac{T}{2}}}{T} \sum_{k=-\infty}^{\infty} (-1)^k X_c(\omega - \frac{2\pi}{T}k)$

let $X_4(\omega) = X_5(\omega) e^{-j\omega\frac{T}{2}}$ (ignore phase)

so $X_5(\omega)$ looks like

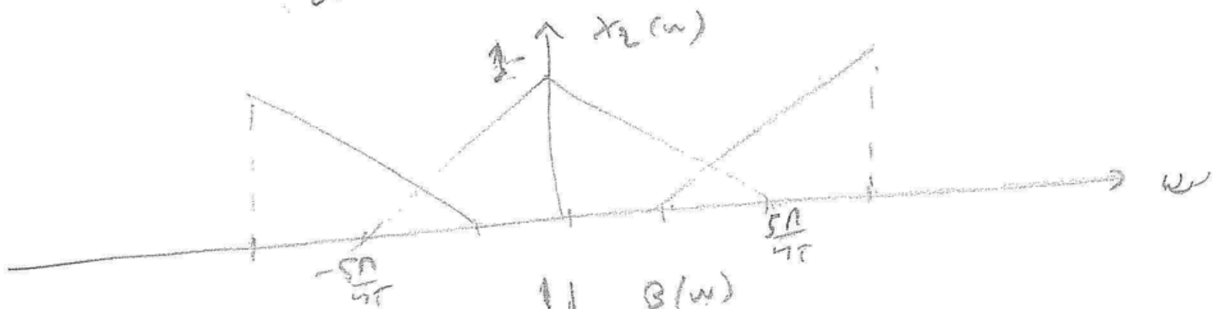


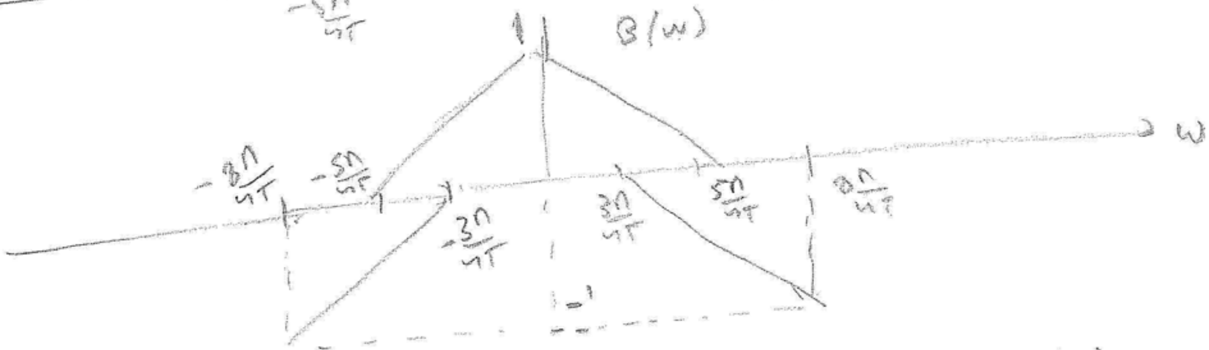
Therefore $Y_2(\omega) = X_5(\omega) \cdot H(\omega) \cdot e^{-j\omega\frac{T}{2}}$

Also $Y_1(\omega) = X_1(\omega) \cdot H(\omega) \cdot e^{-j\omega\frac{T}{2}}$

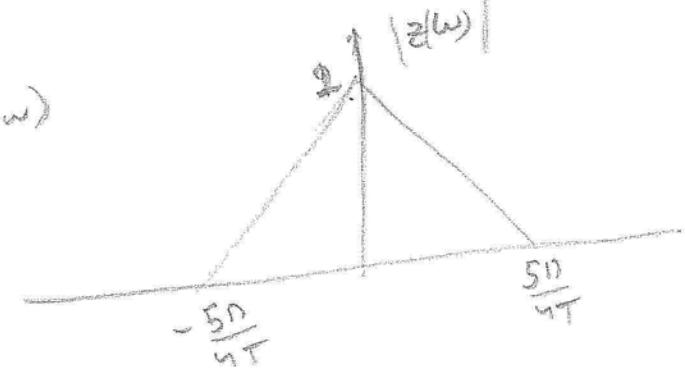
$$Z(\omega) = Y_1(\omega) + Y_2(\omega)$$

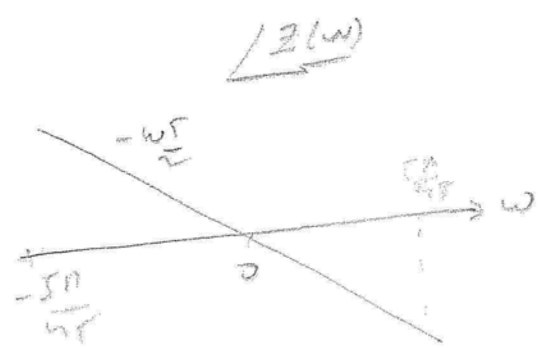
$$= \Big[\underbrace{X_1(\omega)\, H(\omega)}_{X_2(\omega)} + \underbrace{X_5(\omega)\, H(\omega)}_{B(\omega)}\Big] e^{-j\omega\frac{T}{2}}$$





$\therefore Z(\omega)$





$\therefore z_c(t) = 2x_c(t - \frac{T}{2})$ $\boxed{d = 2}$

(12)