Final Exam: MATH 212 (Introductory PDEs)

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Duration: 120 minutes

Name (Last, First): ________________________________

Student number: ________________________________

Circle your instructor's name and your section's number:

M. El Smaily Section: 1 (from 11:00 am to 12:00 pm)

W. Mahboub Section: 2 (from 12:00 pm to 1:00pm),

Section 3 (from 1:00pm to 2:00pm)

For marker's use only	
Problem	Score
1	/15
2	/15
3	/10
4	/12
5	/13
6	/ 12
7	/13
8	/10
Total	/100

[15 points=5+7+3] Problem 1. The goal of this problem is to solve the PDE

$$u_x - 2u_y - 5u = 2x^2 - 3xy - 2y^2$$

with the initial condition $u(x, 0) = e^x$. We will proceed as follows:

(a) Let $t = x - 2y$ and $s = 2x + y$ and $v(t, s) = u(x, y)$. Use the PDE satisfied by $u(x, y)$ and the chain rule to show that v satisfies the PDE

$$5v_t - 5v = st.$$

(b) Use e^{-t} as an integrating factor to solve the PDE obtained in part (a) above.

(c) Deduce $u(x, y)$ interms of x and y (use the initial conditions in the process).

[15 points=6+6+3] Problem 2. Let

$$f(x) = \begin{cases} 1 & \text{for} \quad -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{for} \quad -2 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $f * g(x)$.

(b) Find the Fourier transform $\widehat{f}(k)$ and $\widehat{g}(k)$.

(c) Find the Fourier transform $\widehat{f * g}(k)$ using a method of your choice.

[10 points=5+5] Problem 3. Consider the PDE

$$u_t = u_{xx} + t \sin x, \quad u_x(t, 0) = -1, \quad u_x(t, \pi) = 0 \quad u(0; x) = \cos x.$$

Let $E(t) = \int_0^\pi u(t, x) dx$ denote the total thermal energy at time t.

(a) Compute $\frac{dE(t)}{dt}$ in terms of t

(b) Conclude $E(t)$ in terms of t.

[12 points] Problem 4. Solve using a method of your choice the initial value problem

$$u_t + u_x - 3u = t \text{ for } \ t > 0, \quad u(0, x) = x^2.$$

[13 points=5+5+3] Problem 5. Consider the following boundary value problem

$$\left\| \begin{array}{l} u_{xx}(x,y) + u_{yy}(x,y) - 16u(x,y) + 8(u_x + 4u) = 0, \quad 0 \le x \le 1, \ 0 \le y \le 1, \\ u(x,0) = 2\sin 3x, \\ u(x,1) = 0, \quad u(1,y) = u(0,y) = 0. \end{array} \right.$$

(a) Let $v(x,y) = e^{4x}u(x,y)$. Show that $v(x,y)$ verify a planar Laplace equation (L) with suitable boundary conditions.

(b) Write $v(x, y)$ in a series format–compute the coefficients of the trigonometric terms.

(c) Deduce a series format for $u(x, y)$.

[12 points=5+7] Problem 6. Consider the heat problem with **inhomogeneous** Dirichlet boundary conditions

$$\left\| \begin{array}{l} u_t(t,x) = u_{xx}(t,x) \text{ for } \ 0 \leq x \leq 2, \ t \geq 0, \\ u(t,0) = 0, \quad u(t,2) = 2, \end{array} \right.$$

and the initial datum

$$u(0,x) = \begin{cases} x \text{ if } \ 0 \leq x \leq 1 \\ 2 - x \text{ if } 1 < x \leq 2 \end{cases}$$

(a) Find the equilibrium state $u^*(x)$.

(b) Using the function $v(t, x) = u(t, x) - u^*(x)$, find $u(x, y)$.

[13 points=4+6+3] Problem 7. Consider the sequences of functions $\{f_n(x)\}$ and $\{g_n(x)\}$ defined by

$$f_n(x) = \frac{n}{nx+1}, \quad x \in (0,1)$$

and

$$g_n(x) = \left(1 + \frac{x}{n}\right)^n, \quad x \in \mathbb{R}.$$

(a) Find the pointwise limits f and g of $\{f_n(x)\}$ and $\{g_n(x)\}$ respectively.

(b) Show that $\{f_n(x)\}$ converges to f uniformly on $[a, 1)$ for every $0 < a < 1$ but not over $(0, 1)$.

(c) Does $\{g_n(x)\}$ converge uniformly to g over $\mathbb{R}$? Justify your answer.

[10 points] Problem 8. Show that, if $v(t,x) > 0$ is any positive solution to the linear heat equation $v_t = \gamma v_{xx}$, then $u(t,x) = \dfrac{\partial}{\partial x}\left(-2\gamma \log v(t,x)\right)$ solves Burgers' equation $u_t + u u_x = u_{xx}$.