American University of Beirut
Faculty of Arts and Science
Quiz 2 — MATH 212 — Summer 2015

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Duration: 35 minutes

Last Name: Solutions

First Name:

Student number:

No Calculators or other aids are allowed
(cell phones must be turned off during this exam)
No credit will be given if the answers are not justified.
No books, no course notes, no cheat sheets are allowed

For marker's use only	
Question	Mark
1	/25
2	/15
3	/10
Total	/50

[25 points=5+10+5+5] **Problem 1.** Consider the function $f(x) = x(\pi - x)$ for $0 \le x \le \pi$.

(a) Graph the extension $g(x)$ of $f(x)$ to the real-line such that g is even and 2π-periodic over the real line $\mathbb{R}$.

(b) Find the Fourier series of g.

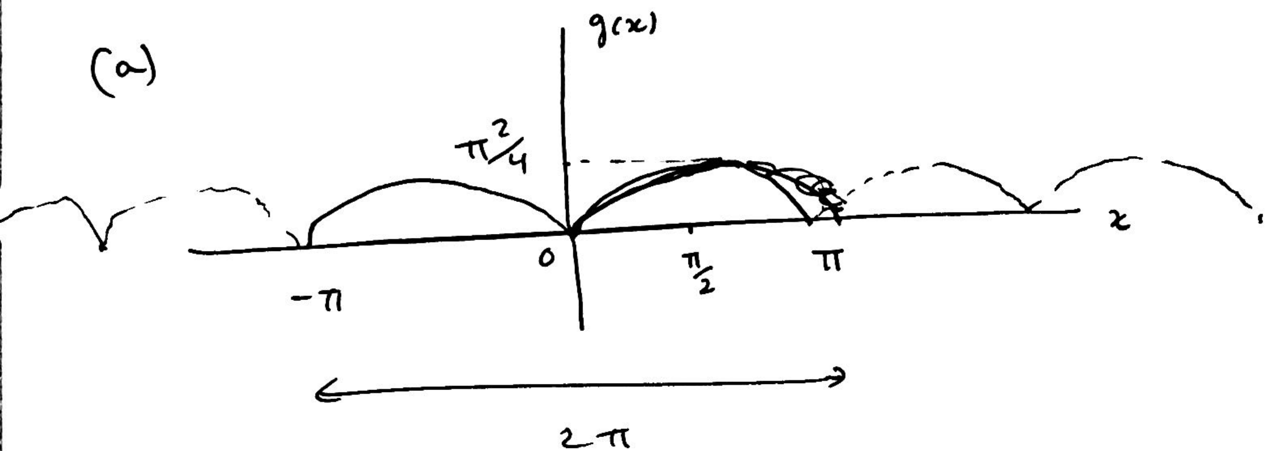
(c) Explain why it is true that, *for all* $0 \le x \le \pi$, we have

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right) ?$$

Mainly why is the above equality true for $x = 0$ and $x = \pi$?

(d) Use the preceding parts to show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

(a)



(b) g is even. So, $a_n = \frac{2}{l}\int_0^l g(x) \cos\frac{n\pi x}{l}\,dx$

$b_n = 0$ for all n.

Here, $2l = 2\pi$. So $l = \pi$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(x) \cos nx \, dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\pi \cdot \frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi} \cdot \frac{\pi^3}{6} = \frac{\pi^2}{3}$$

$$\boxed{\frac{a_0}{2} = \frac{\pi^2}{6}}$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^\pi \underbrace{x(\pi - x)}_{u}\ \underbrace{\cos nx\, dx}_{dv}$$

$u = \pi x - x^2$

$$du = (\pi - 2x)\,dx$$

Integrate by parts

$$a_n = \frac{2}{\pi}\left[\overset{=0}{\frac{(\pi - x)x}{n}\sin nx\Big|_0^\pi} - \frac{1}{n}\int_0^\pi (\sin nx)(\pi - 2x)\,dx \right]$$

$$= -\frac{2}{\pi n}\cdot\pi\int_0^\pi \sin nx\,dx + \frac{4}{\pi n}\int_0^\pi \underbrace{x}_{u}\ \underbrace{\sin nx\,dx}_{dv}$$

Integrate by parts

$$= +\frac{2}{n}\cdot\frac{1}{n}\cos nx\Big|_0^\pi + \frac{4}{\pi n}\left[-\frac{x}{n}\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right]$$

$$= \frac{2}{n^2}\left((-1)^n - 1\right) + \frac{4}{\pi n}\left[-\frac{\pi}{n}(-1)^n - 0 + \underbrace{\frac{1}{n^2}\sin nx\Big|_0^\pi}_{=0}\right]$$

$$= \left(\frac{2-4}{n^2}\right)(-1)^n - \frac{2}{n^2} = -2\left(\frac{(-1)^n + 1}{n^2}\right)$$

$$a_n = \begin{cases} \dfrac{-4}{n^2} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$b_n = 0 \ \forall n$

as g is even.

Therefore, the Fourier Series of $g(x)$

is $FS(g(x)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$= \frac{\pi^2}{6} + \sum_{n=2}^{\infty} a_n \cos nx \qquad (a_1 = 0)$$

$$= \frac{\pi^2}{6} + \sum_{k=1}^{\infty} a_{2k} \cos 2kx \qquad (a_{2k+1} = 0)$$

$n = 2k$

$$= \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \frac{-4}{4k^2} \cos 2kx .$$

$$\boxed{FS(g(x)) = \frac{\pi^2}{6} \cancel{+} \sum_{k=1}^{\infty} \frac{\cos 2kx}{k^2}}$$

b) We see that $g(x)$ is piecewise C^1 on $\mathbb{R}$, g is 2π-periodic.

~~Furthermore~~ Thus, at each x, we have

$$\frac{g(x^+) + g(x^-)}{2} = FS(g(x)).$$

However, g is also continuous at each x (see graph).

This makes $FS(g(x)) = \frac{g(x) + g(x)}{2} = g(x)$

for all $x \in [0, \pi]$.

c) $g(\frac{\pi}{2}) = \frac{\pi^2}{4} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{4} - \pi^2/6 = \frac{\pi^2}{12}$.

[15 points=5+5+5] **Problem 2.**

(a) State Bessel's inequality for a real-valued square integrable 2π-periodic function $f(x)$.

(b) Compute $\|g\|_{L^2(-\pi,\pi)}$ where $g(x)$ is the even function over $[-\pi,\pi]$ such that, for $0 \le x \le \pi$, $g(x) = x(\pi - x)$.

(c) Conclude, using part (c) in Problem 1, that

$$\sum_{n=1}^{\infty} \frac{(1)}{n^4} \le \frac{\pi^4}{90}$$

(a) Bessel's Inequality:

Let f be a square-integrable, function over $(-\pi, \pi)$. Denote by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

or $$\sum_{n=-\infty}^{\infty} c_n e^{-inx}$$

the Fourier series (Real or complex) of the 2π-periodic extension of f.

Then, $$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \|f\|_{L^2}^2$$ or

equivalently $$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \le \|f\|_{L^2}^2.$$

(b) $$\|g\|_{L^2}^2 \; \boxed{g \text{ is even}} = \frac{1}{\pi}\int_{-\pi}^{\pi} (g(x))^2 dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\, dx = \cdots$$

$$= \frac{\pi^4}{15}.$$

(c) $a_0 = \frac{\pi^2}{3}$

$\frac{a_0^2}{2} = \frac{\pi^4}{18}$.

for g, $a_n = \begin{cases} \frac{-1}{K^2}, & n = 2k \\ 0 \end{cases}$

$\|g\|_{L^2}^2 = \frac{\pi^4}{15}$.

$b_n = 0 \ \forall n$

By Bessel's ineq, we have

$$\frac{\pi^4}{18} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{\pi^4}{15}$$

$$\sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{k=1}^{\infty} \left(\frac{-1}{k^2}\right)^2 = \sum_{k=1}^{\infty} \frac{1}{k^4}$$

$a_n = 0$ when n is odd.

So $$\sum_{k=1}^{\infty} \frac{1}{k^4} \leq \underbrace{\frac{\pi^4}{15} - \frac{\pi^4}{18}}$$

$$= \frac{\pi^4}{90}.$$

[10 points=5+5] **Problem 3.**

(a) Find, without integral computations, the complex Fourier series of the 2π-periodic function $f(x) = \sin^3 x$.

(b) What is the Fourier series (real) of the 8π-periodic function

$$g(x) = 250\cos\frac{3x}{2} + 2\sin x\cos x.$$

(a) We know that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ for all $x \in (-\pi, \pi)$.

So $\sin^3 x = \dfrac{1}{(2i)^3}\left(e^{3ix} - 3e^{2ix}\cdot e^{-ix} + 3e^{ix}\cdot e^{-2ix} - e^{-3ix}\right)$

$$= \frac{-1}{8i}\left(e^{3ix} - 3e^{ix} - 3e^{-ix} - e^{-3ix}\right)$$

$$= \frac{i}{8}\left(-e^{-3ix} - 3e^{-ix} - 3e^{+ix} + e^{3ix}\right).$$

The above is the (2π-periodic) Complex Fourier representation of $\sin^3 x$.

(b) The FS (8π-periodic) must be of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{4\pi} + b_n\sin\frac{n\pi x}{4\pi} \qquad \text{as } 8\pi = 2l$$

i.e $\cos\frac{nx}{4}$, $\sin\frac{nx}{4}$ are the trig. components.

However, $\cos\frac{3x}{2} = \cos\frac{6x}{4}$ & $2\sin x\cos x = \sin 2x = \sin\frac{8x}{4}$

So, $g(x) = 250\cos\frac{6x}{4} + \sin\frac{8x}{4}$.

This implies that $a_6 = 250$ & $b_8 = 1$ & $a_n = 0$ for $n \neq 6$, $b_n = 0$ for $n \neq 8$