Final Exam : MATH 212 (Introductory PDEs)

Instructor: Wael Mahboub

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Duration: 120 minutes

Name (Last, First): ________________________________

Student number: ________________________________

For marker's use only	
Problem	Score
1	/10
2	/20
3	/10
4	/15
5	/15
6	/ 20
7	/ 10
Total	/100

[10 points=5+5] Problem 1. Consider the following PDE

$$u_t = u_{xx} + 2u_x - 3u.$$

(a) Let $v(t,x) = e^{3t}u(t,x)$. Use the PDE satisfied by $u(x,y)$ to show that v satisfies the PDE.

$$v_t + 2v_x = v_{xx}. \tag{1}$$

(b) Let $\xi = x - 2t$, and $W(t,\xi) = v(t,x)$. Use the PDE satisfied by $v(t,x)$ and chain rule to show that W satisfies a heat equation.

[25 points=3+2+5+5+5] Problem 2. Consider the following boundary value problem

$$u_{xx} + u_{yy} - u = 0, \quad u(0, y) = u(\pi, y) = u(x, 0) = 0, \quad u(x, \pi) = f(x).$$

(a) Show that separating as $u(x, y) = v(x).w(y)$, we get a seperation constant λ and two ordinary differential equations

$$v'' - v - \lambda v = 0. \tag{2}$$

$$w'' + \lambda w = 0 \tag{3}$$

(b) Write the boundary conditions that v and w must satisfy.

(c) Show that the ODE (2) admits a non trivial solution, if and only if $\lambda = -1 - n^2$ where n is a integer.

(d) Show that for each positive integer n, for $\lambda = -1 - n^2$, the ODE (3) has a solution involving the hyperbolic sine function (Hint: Use $w(0) = 0$).

(e) Conclude a series solution of the form $u(x, y) = \sum_{n=1}^{\infty} c_n u_n(x, y)$ where the constants c_n must be computed in terms of the function f.

[10 points] Problem 3. Solve the following boundary value problem.

$$\left\| \begin{array}{l} \Delta u = 0, \quad x^2 + y^2 < 1, \\ u(x, y) = x, \ x^2 + y^2 = 1. \end{array} \right.$$

. Where Δ is the Laplace Operator
(You're not required to give a proof of your formulas).

[15 points=10+5] Problem 4. Let D be the square $\{(x,y) \in \mathbb{R}^2 /\ 0 \leq x < \pi,\ 0 \leq y < \pi\}$.

1. Solve the heat problem

$$\left\| \begin{array}{l} u_t = u_{xx} + u_{yy}, \quad (x,y) \in D, \\ u(t,0,y) = u(t,\pi,y) = u(t,x,0) = u(t,x,\pi) = 0, \\ u(0,x,y) = f(x,y), t \geq 0. \end{array} \right.$$

   Where f is a continuous function in D.
   (You're not required to give a proof of your formulas).

2. Give the solution in the case $f(x, y) = 2sin(x)sin(2y) - 3sin(2x)sin(5y)$.

[15 points=6+6+3] Problem 5. Let

$$f(x) = \begin{cases} 2 & \text{for} \quad -3 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the Fourier transform $\widehat{f}(k)$.

(b) Find $f * f(x)$.

(c) Find the Fourier transform $\widehat{f * f}(k)$.

[20 points=6+7+7] Problem 6. Consider the following heat problem.

$$\left\|\begin{array}{l} u_t = u_{xx} - 2tu, \quad (x \in \mathbb{R},\ t \geq 0), \\ u(0,x) = e^{-x^2}, \\ u(t,x) \to 0 \ as\ |x| \to \infty,\ uniformly\ in\ t. \end{array}\right.$$

1. Show that the Fourier transform of e^{-x^2} is given by $\widehat{e^{-x^2}} = \frac{e^{\frac{-k^2}{4}}}{\sqrt{2}}$.
(Hint: Use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$).

2. Let $\hat{u}$ denote the Fourier transform of u with respect to x. Show that

$$\hat{u}(t,k) = \frac{e^{-k^2(t+\frac{1}{4})-t^2}}{\sqrt{2}}.$$

3. Show that the solution is given by

$$u(t,x) = \frac{e^{-t^2}}{\sqrt{\pi}} \int_0^\infty e^{-k^2(t+\frac{1}{4})} cos(kx) dx.$$

[10 points=6+4] Problem 7. Let $f(x) = \sin x$ for $0 \le x < \pi$.

(a) Expand the function $f(x) = \sin x$ in a Fourier cosine series in the range $0 \le x < \pi$.

(b) Use the series to evaluate the sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$.