MATH 212: INTRODUCTORY PARTIAL DIFFERENTIAL EQUATIONS

SOPHIE MOUFAWAD

Exercise 3.4.8 a) Similarly to the case where $f(x)$ is defined over $[-\pi, \pi]$, in case $f(x)$ is defined over $[0, 2\pi]$, we have that:

$$\int_0^{2\pi} sin(lx)cos(kx)dx = 0 \tag{0.1}$$

$$\int_0^{2\pi} sin(lx)sin(kx)dx = \begin{cases} 0 & if \;\; k \neq l \\ \pi & if \;\; k = l \end{cases} \tag{0.2}$$

$$\int_0^{2\pi} cos(lx)cos(kx)dx = \begin{cases} 0 & if \;\; k \neq l \\ \pi & if \;\; k = l \end{cases} \tag{0.3}$$

$$\int_0^{2\pi} cos(kx)dx = 0 \tag{0.4}$$

$$\int_0^{2\pi} sin(kx)dx = 0 \tag{0.5}$$

where $k, l \in \mathbb{N}$, and $k > 0$ and $l > 0$.

Thus, we can write the Fourier series of a function $f(x)$ defined over $0 \leqslant x \leqslant 2\pi$, denoted by $FS[f(x)]$ as

$$FS[f(x)] = a_0/2 + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx) \tag{0.6}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, $a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)cos(kx)dx$, and $b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(kx)dx$ for $k = 1, 2, ...$

To get the coefficients a_0, a_k, and b_k, we assume that the Fourier series converges to $f(x)$, and use the above integrals. Then

$$f(x) = FS[f(x)] = a_0/2 + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx) \tag{0.7}$$

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi} a_0/2dx + \sum_{k=1}^{\infty} \int_0^{2\pi} (a_k cos(kx) + b_k sin(kx))dx \tag{0.8}$$

$$\int_0^{2\pi} f(x)dx = \pi a_0 \tag{0.9}$$

For $l \in \mathbb{N}$,

$$\frac{1}{\pi}\int_0^{2\pi} f(x)cos(lx)dx = \frac{a_0}{2\pi}\int_0^{2\pi} cos(lx)dx + \sum_{k=1}^{\infty}[\frac{a_k}{\pi}\int_0^{2\pi} cos(kx)cos(lx)dx + \frac{b_k}{\pi}\int_0^{2\pi} sin(kx)cos(lx)dx]$$

$$\frac{1}{\pi}\int_0^{2\pi} f(x)cos(kx)dx = \frac{a_k}{\pi}\int_0^{2\pi} cos(kx)cos(kx)dx = a_k. \tag{0.10}$$

Similarly, for $l \in \mathbb{N}$, we have that

$$\frac{1}{\pi}\int_0^{2\pi} f(x)sin(lx)dx = \frac{a_0}{2\pi}\int_0^{2\pi} sin(lx)dx + \sum_{k=1}^{\infty}[\frac{a_k}{\pi}\int_0^{2\pi} cos(kx)sin(lx)dx + \frac{b_k}{\pi}\int_0^{2\pi} sin(kx)sin(lx)dx]$$

$$\frac{1}{\pi}\int_0^{2\pi} f(x)sin(kx)dx = \frac{b_k}{\pi}\int_0^{2\pi} sin(kx)sin(kx)dx = b_k. \tag{0.11}$$