MATH 212: INTRODUCTORY PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 7

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Exercise 1: The goal of this exercise is to solve the following heat equation with homogeneous boundary conditions and constant initial condition:

$$\begin{cases} u_t = u_{xx} + \frac{\pi^2}{4}u - b, & t > 0, b \in \mathbb{R} \\ u(t,0) = u(t,1) = 0 & t \geqslant 0 \\ u(0,x) = c, & 0 < x < 1, c \in \mathbb{R}. \end{cases}$$

1. Solve the ODE $z''(x) + \frac{\pi^2}{4}z(x) = b$.

   Solution to the homogeneous ODE $z''(x) + \frac{\pi^2}{4}z(x) = 0$ is $z_h(x) = c_1 cos(\pi x/2) + c_2 sin(\pi x/2)$. A particular solution is $z_p(z) = \frac{4b}{\pi^2}$. Then, the solution is $z(x) = z_h(x) + z_p(x) = c_1 cos(\pi x/2) + c_2 sin(\pi x/2) + \frac{4b}{\pi^2}$
2. Find the equilibrium solution $u_E(x)$.

   $$\begin{cases} (u_E)_t = 0 = (u_E)_{xx} + \frac{\pi^2}{4}u_E - b, & b \in \mathbb{R} \\ u_E(t,0) = u_E(t,1) = 0 & t \geqslant 0 \end{cases}$$

   Since $(u_E)_t = 0$, then $u_E(t,x) = u_E(x)$ and the pde is transformed into an ODE with boundary conditions:

   $$\begin{cases} (u_E)'' + \frac{\pi^2}{4}u_E = b, & b \in \mathbb{R} \\ u_E(0) = u_E(1) = 0 \end{cases}$$

   The solution to the ODE is $u_E(x) = c_1 cos(\pi x/2) + c_2 sin(\pi x/2) + \frac{4b}{\pi^2}$. Then, $u_E(0) = c_1 + \frac{4b}{\pi^2} = 0$ and $c_1 = -\frac{4b}{\pi^2}$. Similarly, $u_E(1) = -\frac{4b}{\pi^2}cos(\pi/2) + c_2 sin(\pi/2) + \frac{4b}{\pi^2} = 0$. Then $c_2 = -\frac{4b}{\pi^2}$.

   $u_E(x) = -\frac{4b}{\pi^2}cos(\pi x/2) - \frac{4b}{\pi^2}sin(\pi x/2) + \frac{4b}{\pi^2} = \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))$
3. Define the difference $v(t,x) = u(t,x) - u_E(x)$ that measures the deviation of the solution $u(t,x)$ from equilibrium.
   (a) Show that $v(t,x)$ satisfies a heat equation of the form $v_t = \gamma v_{xx} + \beta v$, where $\beta, \gamma \in \mathbb{R}$.

   $u(t,x) = v(t,x) + u_E(x)$. Then $v_t = u_t$, $u_{xx} = v_{xx} + (u_E)''$. But, $u(t,x)$

satisfies $u_t = u_{xx} + \frac{\pi^2}{4}u - b$. Then $v_t = v_{xx} + (u_E)'' + \frac{\pi^2}{4}(v + u_E) - b =$ $v_{xx} + \frac{\pi^2}{4}v + (u_E'' + \frac{\pi^2}{4}u_E) - b$.

But $(u_E'' + \frac{\pi^2}{4}u_E) = b$. Thus, $v_t = v_{xx} + \frac{\pi^2}{4}v$, where $\gamma = 1$, and $\beta = \frac{\pi^2}{4}$.

(b) Then, adapt the boundary and initial value conditions.

$v(t, 0) = u(t, 0) - u_E(0) = 0 - 0 = 0$, $v(t, 1) = u(t, 1) - u_E(1) = 0 - 0 = 0$.

$v(0, x) = u(0, x) - u_E(x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))$. Then

$$\begin{cases} v_t = v_{xx} + \frac{\pi^2}{4}v, & t > 0 \\ v(t,0) = v(t,1) = 0 & t \geqslant 0 \\ v(0,x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2)), & 0 < x < 1, c \in \mathbb{R}. \end{cases}$$

4. There are two ways for solving initial-boundary value problem obtained in part 3.

(a) Let $w(t, x) = e^{\alpha t}v(t, x)$. Find an α value for which $w(t, x)$ satisfies the heat equation $w_t = w_{xx}$. Then, change the boundary and initial conditions accordingly. And solve the obtained initial boundary value problem. Finally, deduce the solution $v(t, x)$.

Solution

$w_t = \alpha e^{\alpha t}v + e^{\alpha t}v_t$, and $w_{xx} = e^{\alpha t}v_{xx}$. Then, $\alpha e^{\alpha t}v + e^{\alpha t}v_t = e^{\alpha t}v_{xx}$. Moreover, $v_t = v_{xx} - \alpha v$. But $v(t, x)$ solves $v_t = v_{xx} + \frac{\pi^2}{4}v$. Then, $\alpha = -\frac{\pi^2}{4}$, and $w(t, x) = e^{-\frac{\pi^2}{4}t}v(t, x)$.

Or alternatively, $v(t, x) = e^{-\alpha t}w(t, x)$, $v_t = -\alpha e^{-\alpha t}w + e^{-\alpha t}w_t$, and $v_{xx} =$ $e^{-\alpha t}w_{xx}$. Thus, $-\alpha e^{-\alpha t}w + e^{-\alpha t}w_t = e^{-\alpha t}w_{xx} + \frac{\pi^2}{4}e^{-\alpha t}w$, and $w_t =$ $w_{xx} + (\frac{\pi^2}{4} + \alpha)w$. Thus, $\alpha = -\frac{\pi^2}{4}$.

$v(t, 0) = e^{-\alpha t}w(t, 0) = 0 = e^{-\alpha t}w(t, 1) = v(t, 1)$, then $w(t, 0) = w(t, 1) =$ 0. $w(0, x) = e^0 v(0, x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))$.

$$\begin{cases} w_t = w_{xx}, & t > 0 \\ w(t,0) = w(t,1) = 0 & t \geqslant 0 \\ w(0,x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2)), & 0 < x < 1, c \in \mathbb{R}. \end{cases}$$

By the method of separation of variables, we get that $w(t, x) = \sum_{k=1}^{\infty} b_k e^{-(k\pi)^2 t} sin(k\pi x) =$ $\sum_{k=1}^{\infty} w_k(t, x)$ where $b_k = 2\int_0^1 [c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))] sin(k\pi x)dx$.

Using the trigonometric identities $sin(\alpha)cos(\beta) = \frac{1}{2}(sin(\alpha + \beta) + sin(\alpha - \beta))$

and $sin(\alpha)sin(\beta) = \frac{1}{2}(cos(\alpha-\beta) - cos(\alpha+\beta))$ we get

$$
\begin{aligned}
b_k &= (2c - \frac{8b}{\pi^2})\int_0^1 sin(k\pi x)dx + \frac{8b}{\pi^2}\int_0^1 (cos(\pi x/2) + sin(\pi x/2))sin(k\pi x)dx \\
&= (-2c + \frac{8b}{\pi^2})\frac{(-1)^k - 1}{k\pi} + \frac{4b}{\pi^2}\int_0^1 [sin(\frac{2k+1}{2}\pi x) + sin(\frac{2k-1}{2}\pi x) + cos(\frac{2k-1}{2}\pi x) - cos(\frac{2k+1}{2}\pi x)]dx \\
&= (-2c + \frac{8b}{\pi^2})\frac{(-1)^k - 1}{k\pi} + \frac{4b}{\pi^2}[-2\frac{cos(\frac{2k+1}{2}\pi x)}{(2k+1)\pi} - 2\frac{cos(\frac{2k-1}{2}\pi x)}{(2k-1)\pi} + 2\frac{sin(\frac{2k-1}{2}\pi x)}{(2k-1)\pi} - 2\frac{sin(\frac{2k+1}{2}\pi x)}{(2k+1)\pi}]_0^1 \\
&= (-2c + \frac{8b}{\pi^2})\frac{(-1)^k - 1}{k\pi} + \frac{8b}{\pi^3}[\frac{sin(\frac{2k-1}{2}\pi)}{(2k-1)} - \frac{sin(\frac{2k+1}{2}\pi)}{2k+1} + \frac{1}{2k+1} + \frac{1}{2k-1}] \\
&= (-2c + \frac{8b}{\pi^2})\frac{(-1)^k - 1}{k\pi} + \frac{8b}{\pi^3}[\frac{(-1)^{k+1}+1}{2k-1} + \frac{(-1)^{k+1}+1}{2k+1}] \\
&= (-2c + \frac{8b}{\pi^2})\frac{(-1)^k - 1}{k\pi} + \frac{32bk}{(2k-1)(2k+1)\pi^3}[(-1)^{k+1}+1] \\
&= \begin{cases} 0, & k = 2j \\ \frac{4c}{(2j-1)\pi} - \frac{16b}{(2j-1)\pi^3} + \frac{32b(2j-1)}{(4j-3)(4j-1)\pi^3}, & k = 2j-1 \end{cases}
\end{aligned}
$$

Then, $w_2(t,x) = w_4(t,x) = w_{2j}(t,x) = 0$, and

$$
w(t,x) = \sum_{k=1}^{\infty} w_k(t,x) = \sum_{j=1}^{\infty}[\frac{4c}{(2j-1)\pi} - \frac{16b}{(2j-1)\pi^3} + \frac{32b(2j-1)}{(4j-3)(4j-1)\pi^3}]e^{-(2j-1)^2\pi^2 t}sin((2j-1)\pi x)
$$

$$
\begin{aligned}
v(t,x) &= e^{\frac{\pi^2}{4}t}w(t,x) = e^{\frac{\pi^2}{4}t}\sum_{k=1}^{\infty} w_k(t,x) \\
&= e^{\frac{\pi^2}{4}t}\sum_{j=1}^{\infty}[\frac{4c}{(2j-1)\pi} - \frac{16b}{(2j-1)\pi^3} + \frac{32b(2j-1)}{(4j-3)(4j-1)\pi^3}]e^{-(2j-1)^2\pi^2 t}sin((2j-1)\pi x)
\end{aligned}
$$

(b) Solve for $v(t,x)$ directly by using the method of separation of variables.

$$
\begin{cases} v_t = v_{xx} + \frac{\pi^2}{4}v, & t > 0 \\ v(t,0) = v(t,1) = 0 & t \geqslant 0 \\ v(0,x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2)), & 0 < x < 1, c \in \mathbb{R}. \end{cases}
$$

$v(t,x) = S(t)V(x)$, $S'(t)V(x) = S(t)V''(x) + \frac{\pi^2}{4}S(t)V(x)$, then

$$
\frac{S'(t)}{S(t)} = \frac{V''(x) + \frac{\pi^2}{4}V(x)}{V(x)} = \lambda
$$

$S(t) = e^{\lambda t}$ and the solution to $V''(x) = (\lambda - \frac{\pi^2}{4})V(x)$ is

$$
V(x) = \begin{cases} c_1 e^{\omega x} + c_2 e^{-\omega x}, & if \ \ (\lambda - \frac{\pi^2}{4}) = \omega^2 > 0 \\ c_1 sin(\omega x) + c_2 cos(\omega x), & if \ \ (\lambda - \frac{\pi^2}{4}) = -\omega^2 < 0 \\ c_1 + c_2 x, & if \ \ (\lambda - \frac{\pi^2}{4}) = 0 \end{cases}
$$

The first and the last cases are discarded, and $V(x) = c_1 sin(\omega x) + c_2 cos(\omega x)$, where $V(0) = c_2 = 0$, and $V(1) = c_1 sin(\omega) = 0$, implies $\omega = k\pi$, and $\lambda - \frac{\pi^2}{4} = -\omega^2 = -k^2\pi^2$ and $\lambda = \frac{\pi^2}{4} - k^2\pi^2$.
Then the eigensolutions are, $v_k(t,x) = e^{\frac{\pi^2}{4}t - k^2\pi^2 t} sin(k\pi x)$ for $k = 1, 2, 3, ...$

$$v(t,x) = e^{\frac{\pi^2}{4}t} \sum_{k=1}^{\infty} b_k e^{-k^2\pi^2 t} sin(k\pi x)$$

$v(0,x) = \sum_{k=1}^{\infty} b_k sin(k\pi x) = c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))$, and

$$\begin{aligned} b_k &= 2\int_0^1 [c - \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))] sin(k\pi x) dx \\ &= \begin{cases} 0, & k = 2j \\ \frac{4c}{(2j-1)\pi} - \frac{16b}{(2j-1)\pi^3} + \frac{32b(2j-1)}{(4j-3)(4j-1)\pi^3}, & k = 2j-1 \end{cases} \end{aligned}$$

And $v(t,x)$ has the same expression as in part a.

5. Deduce the solution $u(t,x)$ and find its limit as t tends to infinity.

$$\begin{aligned} u(t,x) &= u_E(x) + v(t,x) \\ &= \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2)) \\ &\quad + e^{\frac{\pi^2}{4}t} \sum_{j=1}^{\infty} [\frac{4c}{(2j-1)\pi} - \frac{16b}{(2j-1)\pi^3} + \frac{32b(2j-1)}{(4j-3)(4j-1)\pi^3}] e^{-(2j-1)^2\pi^2 t} sin((2j-1)\pi x) \end{aligned}$$

$lim_{t\to\infty} u(t,x) = u_E(x) = \frac{4b}{\pi^2}(1 - cos(\pi x/2) - sin(\pi x/2))$
since $|v(t,x)| \leqslant \sum_{k=1}^{\infty} |b_k| e^{\frac{\pi^2}{4}t - k^2\pi^2 t}$, and b_k is bounded

$$\begin{aligned} |b_k| &\leqslant 2\int_0^1 |c - \frac{4b}{\pi^2}| + \frac{4|b|}{\pi^2} cos(\pi x/2) + \frac{4|b|}{\pi^2} sin(\pi x/2) dx \\ &\leqslant 2[|c - \frac{4b}{\pi^2}| + \frac{8|b|}{\pi^3} - \frac{8|b|}{\pi^3}(-1)] \\ &\leqslant 2[|c - \frac{4b}{\pi^2}| + \frac{16|b|}{\pi^3}] \end{aligned}$$

Moreover, $lim_{t\to\infty} e^{\frac{\pi^2}{4}t - k^2\pi^2 t} = 0$. Thus, $lim_{t\to\infty} v(t,x) = 0$.

6. Approximate the solution $u(t,x)$ by $u_E(x)$ and the first term of $v(t,x)$.
   (a) Find an upper bound for the error between $u(t,x)$ and its first term approximation, i.e. $|u(t,x) - u_E(x) - v_1(t,x)|$, when $c = \frac{4|b|}{\pi^2}$.

$$\begin{aligned} |u(t,x) - u_E(x) - v_1(t,x)| = |v(t,x) - v_1(t,x)| &= e^{\frac{\pi^2}{4}t} |w(t,x) - w_1(t,x)| \\ &\leqslant \frac{M e^{\frac{\pi^2}{4}t}}{exp(2\pi^2 t) - exp(\pi^2 t)} \end{aligned} \tag{0.1}$$

where $M = 2\int_0^1 \frac{4|b|}{\pi^2}(cos(\pi x/2)+sin(\pi x/2))dx = \frac{16|b|}{\pi^3}[sin(\pi x/2)-cos(\pi x/2)]_0^1 = \frac{32|b|}{\pi^3} = \frac{8c}{\pi}$. We obtained this upper bound since $w(t,x)$ solves the heat equation with homogeneous boundary condition. Refer to the extra note posted on moodle, where we have derived this upper bound.

But since $w_2(t,x) = 0$, then

$$\begin{aligned}|u(t,x) - u_E(x) - v_1(t,x)| = e^{\frac{\pi^2}{4}t}|w(t,x) - w_1(t,x)| &= e^{\frac{\pi^2}{4}t}|w(t,x) - w_1(t,x) - w_2(t,x)| \\ &\leqslant \frac{Me^{\frac{\pi^2}{4}t}}{exp(3\pi^2 t) - exp(2\pi^2 t)} \\ &\leqslant \frac{8ce^{\frac{\pi^2}{4}t}}{\pi(exp(3\pi^2 t) - exp(2\pi^2 t))}\end{aligned}$$

(b) After what time will the error from the first term approximation be less than $\frac{8ce^{1/4}}{\pi(e^3-e^2)}$.

For $t \geqslant \frac{1}{\pi^2}$ the error is less than $\frac{8ce^{\frac{1}{4}}}{\pi(e^3-e^2)}$

Exercise 2: Consider the following heat equation with mixed boundary conditions and initial condition:

$$\begin{cases} u_t = u_{xx}, & t > 0 \\ u(t,0) = 0 & t \geqslant 0 \\ u_x(t,l) = 0 & t \geqslant 0 \\ u(0,x) = f(x), & 0 < x < l. \end{cases}$$

1. Find the solution to the initial-boundary value problem, and discuss its asymptotic behavior as $t \to \infty$.

   Let $u(t,x) = S(t)V(x)$, $S'(t)V(x) = S(t)V''(x)$, and $\frac{S'(t)}{S(t)} = \frac{V''(x)}{V(x)} = \lambda$. $S(t) = e^{\lambda t}$. And

$$V(x) = \begin{cases} c_1 e^{\omega x} + c_2 e^{-\omega x}, & if \;\; \lambda = \omega^2 > 0 \\ c_1 sin(\omega x) + c_2 cos(\omega x), & if \;\; \lambda = -\omega^2 < 0 \\ c_1 + c_2 x, & if \;\; \lambda = 0 \end{cases}$$

   where $V(0) = 0$ and $V'(l) = 0$.

   If $\lambda = \omega^2 > 0$, then $V(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$, and $V'(x) = \omega c_1 e^{\omega x} - \omega c_2 e^{-\omega x}$. $V(0) = c_1 + c_2 = 0$, $c_1 = -c_2$. $V'(l) = c_2(e^{\omega l} + e^{-\omega l}) = 0$, then $c_1 = c_2 = 0$. Discarded.

If $\lambda = 0$, $c_1 = c_2 = 0$. Discarded.

If $\lambda = -\omega^2 < 0$, then $V(x) = c_1 sin(\omega x) + c_2 cos(\omega x)$. $V(0) = c_2 = 0$ and $V'(x) = \omega c_1 cos(\omega x)$, $V'(l) = \omega c_1 cos(\omega l) = 0$ when $\omega l = -\frac{\pi}{2} + k\pi$ for $k = 1, 2, 3, ...$ Then $\omega = \frac{(2k-1)\pi}{2l}$ and $V_k(x) = sin(\frac{(2k-1)\pi}{2l}x)$ and the eigensolutions are $u_k(t,x) = b_k e^{-(\frac{(2k-1)\pi}{2l})^2 t} sin(\frac{(2k-1)\pi}{2l}x)$ for $k = 1, 2, ...$

The solution is $u(t,x) = \sum_{k=1}^{\infty} u_k(t,x) = \sum_{k=1}^{\infty} b_k e^{-(\frac{(2k-1)\pi}{2l})^2 t} sin(\frac{(2k-1)\pi}{2l}x)$.

Using the initial condition, $u(0,x) = \sum_{k=1}^{\infty} b_k sin(\frac{(2k-1)\pi}{2l}x) = f(x)$.

However, this is not the usual Fourier Sine series of a function $f(x)$ defined over $[0, l]$. But we can find the coefficients b_k, for which the sine series converges to $f(x)$ on the interval $[0, l]$. Note that for some $z \in \mathbb{N}$, and $z > 0$, we have

$$\int_0^l sin(\frac{(2k-1)\pi}{2l}x) sin(\frac{(2z-1)\pi}{2l}x)dx = \frac{1}{2}\int_0^l cos(\frac{(k-z)\pi}{l}x) - cos(\frac{(k+z-1)\pi}{l}x)dx$$
$$= \begin{cases} 0, & if k \neq z \\ \frac{l}{2}, & if k = z \end{cases} \tag{0.2}$$

Then,

$$\frac{2}{l}\int_0^l f(x) sin(\frac{(2z-1)\pi}{2l}x)dx = \frac{2}{l}\sum_{k=1}^{\infty} b_k \int_0^l sin(\frac{(2z-1)\pi}{2l}x) sin(\frac{(2k-1)\pi}{2l}x)dx$$
$$= b_z = b_k \tag{0.3}$$

Thus, $b_k = \frac{2}{l}\int_0^l f(x) sin(\frac{(2k-1)\pi}{2l}x)dx$. Each term in the infinite series is bounded by a damping term $|u_k(t,x)| \leqslant M e^{-(\frac{k\pi}{l})^2 t}$ where $M = \frac{2}{l}\int_0^l |f(x)|dx$, thus $lim_{t\to\infty} u(t,x) = 0$

2. Find the equilibrium solution $u_E(x)$.

$$\begin{cases} (u_E)_t = 0 = (u_E)_{xx} & \\ u_E(t,0) = 0 & t \geqslant 0 \\ (u_E)_x(t,l) = 0 & \end{cases}$$

Since $(u_E)_t = 0$, then $u_E(t,x) = u_E(x)$ and the pde is transformed into an ODE with mixed boundary conditions:

$$\begin{cases} (u_E)'' = 0 \\ u_E(0) = (u_E)'(l) = 0 \end{cases}$$

The $u_E(x) = c_1 x + c_2$, $u_E(0) = c_2 = 0$, $u'_E(l) = c_1 = 0$, then $u_E(x) = 0$.