Quiz 1: MATH 201

Instructors: Mohammad El Smaily & Wael Mahboub

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Duration: 60 minutes

Name (Last, First): Solutions

Student number: //////

Circle your instructor's name and your section's number:

M. El Smaily Section: 1 (from 11:00 am to 12:00 pm)

W. Mahboub Section: 2 (from 12:00 pm to 1:00pm),

Section 3 (from 1:00pm to 2:00pm)

For marker's use only	
Problem	Score
1	/20
2	/15
3	/15
4	/15
5	/20
6	/15
Total	/100

[20 points] **Problem 1.** Find the general solution $u(\cdot,\cdot)$ of the following partial differential equations. Justify your work.

(a) **[10 points]** $\frac{\partial^2 u}{\partial x^2}(t,x) - 16u(t,x) = 2.$

Let's regard t as a constant & 1st solve the Homogeneous ODE $Y''(x) - 16Y = 0$.

The G-S is decided by the roots of $r^2 - 16 = 0$

$r = \mp 4$.

So $Y_1(x) = e^{-4x}$, $Y_2(x) = e^{4x}$ are fundam. sols of ODE.

So, For hom. PDE, $U(t,x) = A_1(t)e^{-4x} + A_2(t)e^{4x}$

P_S: $u^*(t,x) = -\frac{1}{8}$ is a P_S.

$u(t,x) = A_1(t)\, e^{-4x} + A_2(t)\, e^{4x} - \frac{1}{8}$ is the G-S for the Given PDE.

(b) **[10 points]** $\frac{\partial^2 u}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = B(y) \quad \text{for some function } B.$$

Thus $u(x,y) = f(y) + g(x)$

where f, g are random functions.

[15 points] Problem 2.

(a) [10 points] Solve the following initial value problem using the method of characteristics

$$x\,u_t - 4t\,u_x = 0, \quad u(0,x) = \frac{1}{1+x^2}.$$

Characteristic variable:

Let $V(t,x) = (x, -4t)$.

PDE $\longrightarrow$ $V \cdot D_{t,x} u = 0$

Let $(t, x(t))$ be a characteristic curve.

Then $\frac{dx}{dt} = \frac{-4t}{x}$. So, $\int x\,dx = \int -4t\,dt$

i.e $\frac{x^2}{2} = -\frac{4t^2}{2} + k$ (k arbitrary const).

$\Rightarrow x^2 + 4t^2 = k$.

So $u(t,x) = \phi(x^2 + 4t^2)$.

$u(0,x) = \frac{1}{1+x^2} \Longrightarrow \phi(x^2+0) = \frac{1}{1+x^2} \Rightarrow \phi(s) = \frac{1}{1+s}$

$\Rightarrow u(t,x) = \frac{1}{1+x^2+4t^2}$

(b) [5 points] Compute $\lim_{t\to\infty} u(t,x)$

$\lim_{t\to\infty} u(t,x) \overset{x \text{ fixed}}{=} \lim_{t\to\infty} \frac{1}{1+x^2+4t^2} = 0.$

[15 points] Problem 3. Let $a, c \in \mathbb{R}$ and consider the equation $u_t + cu_x - au = 0$ with the initial condition $u(0,x) = f(x)$ for some prescribed function f.

e^{-at}

(a) **[8 points]** Show that the function $v(t,x)$ defined by $v(t,x) = e^{at}u(t,x)$ satisfies a transport to be determined.

$$v_t = -a e^{-at} u + e^{-at} u_t .$$

$$c v_x = c (e^{-at} u_x) .$$

$$\Rightarrow v_t + c v_x = -a e^{-at} u + e^{-at} u_t + c e^{-at} u_x$$

$$= e^{-at} (\underbrace{u_t + c u_x - a u}_{=0})$$

$$= e^{-at} \cdot 0 = 0 .$$

(b) **[7 points]** Solve the differential equation that v satisfies and then deduce a formula for the function $u(t,x)$ in terms of a, c and the function f.

$v_t + c v_x = 0$ is a homogeneous uniform transport eq.

$$v(0,x) = e^{-a0} u(0,x) = f(x) \cdot 1 = f(x) .$$

$$\Rightarrow v(t,x) = f(x - ct) .$$

So $u(t,x) = e^{at} v(t,x)$

$$\boxed{u(t,x) = e^{at} f(x-ct) .}$$

[15 points] **Problem 4.** Use d'Alembert's formula to solve the initial value problem:

$$u_{tt} = 4u_{xx}, \quad u(0,x) = e^x, \quad u_t(0,x) = \cos x.$$

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,d$$

where

$$f(x) = u(0,x) = e^x$$

& $$g(y) = u_t(0,y) = \cos y.$$

So

$$u(t,x) = \frac{e^{x-ct}+e^{x+ct}}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos y\,dy$$

$\boxed{c=2}$

$$= \frac{e^{x-ct}+e^{x+ct}}{2} + \frac{1}{4}\sin(x+2t) - \frac{1}{4}\sin(x-2t)$$

$$= \frac{e^{x-2t}+e^{x+2t}}{2} + \frac{1}{4}\left(\sin(x+2t) - \sin(x-2t)\right.$$

[20 points] Problem 5. Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on the interval $0 \le x \le \pi$ subject to mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$.

Plug in the ansatz

$u(t,x) = e^{\lambda t} v(x)$ in

$u_t = u_{xx}$ to get

$\lambda e^{\lambda t} v_{(x)} = e^{\lambda t} v''$.

So $v''(x) - \lambda v(x) = 0$ (1)

$u(t,0) = e^{\lambda t} v(0) = 0 \Rightarrow v(0) = 0$

$u_x(t,\pi) = 0 \Rightarrow v'(\pi) = 0$

Solve the ODE (1) (find λ, v):

if $\lambda = 0$: $v(x) = Ax + b$

& as $v(0) = 0$ then $b = 0$

$v'(\pi) = 0 \rightarrow A = 0$ $\Big\} \Rightarrow v \equiv 0$

& This leads to a trivial $u(t,x)$.

if $\lambda > 0$: $\lambda = p^2$ for some $p \in \mathbb{R}^*$ $(p \neq 0)$.

2 roots $r_1 = p$, $r_2 = -p$ for $r^2 - p^2 =$

& $v(x) = A e^{px} + B e^{-px}$ $\quad v'(x) = Ape^{px} - Bpe^{-px}$

$v(0) = A + B = 0$, $\quad v'(\pi) = p(Ae^{p\pi} - Be^{-p\pi})$

$\rightarrow B = -A$ $\quad = pA\underbrace{(e^{p\pi} + e^{-p\pi})}_{\neq 0} = 0$

So $A = B = 0$

& So $v \equiv 0 \longrightarrow$ Trivial $u(t,x) \equiv 0$.

The only left case is ~~case~~ $\lambda = -p^2 < 0$

& $v(x) = A\cos px + B\sin px$.

$v'(x) = -Ap\sin px + Bp\cos px$.

$\left.\begin{array}{l} v(0) = A + 0 \\ = 0 \end{array}\right\} \to A = 0$, $v'(\pi) = 0 = 0 + Bp\cos p\pi = 0$

$B \neq 0$
$\Rightarrow \cos p\pi = 0 \Rightarrow p = \frac{2k+1}{2}$, $k = 0, 1, 2, \ldots$

$v_k(x) = B_k \sin\left(\frac{2k+1}{2}x\right) \Rightarrow u(t,x) = \sum_{k=0}^{\infty} e^{-\left(\frac{2k+1}{2}\right)^2 t} \sin\left(\frac{2k+1}{2}\right)$

[15 points] **Problem 6.** Find the general solution of the ordinary differential equation

$$u'' - 3u' = e^x \cos x.$$

$U'' - 3U' = 0$ is the homogeneous version of the given ODE.

Corresponding charcteristic polynomial is

$m^2 - 3m = 0 = m(m-3)$.

roots are $m = 0$, or $m = 3$.

$U_1(x) = e^{0\cdot x} = 1$, $U_2(x) = e^{3x}$ are the fundamental solutions.

So $U(x) = A + Be^{3x}$.

Particular Solution: seek $v(x)$ of the form

$ae^x \cos x + be^x \sin x$.

$v' = ae^x\cos x - ae^x \sin x + be^x \sin x + be^x \cos x$

$= (a+b)e^x \cos x + (b-a)e^x \sin x$

$v''(x) = \cancel{(a+b)e^x\cos x} - (a+b)e^x \sin x + (b-a)e^x \sin x$

$+ \cancel{(b-a)e^x \cos x}$

$= 2be^x \cos x - 2ae^x \sin x$

$v'' - 3v' = e^x \cos x \leftrightarrow e^x\cos x(-b-3a) + e^x \sin x(a-3b) = e^x \cos x$

$\Leftrightarrow \left\{\begin{array}{l} -b-3a = 1 \\ a - 3b = 0 \\ 3a - 9b = 0 \end{array}\right\} \to -10b = 1 \to b = -\frac{1}{10}$, $a = -\frac{3}{10}$.

$v(x) = -\frac{3}{10}e^x \cos x - \frac{1}{10}e^x \sin x$

$\boxed{u(x) = A + Be^{3x} - \frac{1}{10}e^x \sin x - 3/10\, e^x \cos x}$, $A, B \in \mathbb{R}$.