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American University of Beirut

الجامعة الأميركية في بيروت

Midterm Exam: MATH 212

Instructor: Mohammad El Smaily

Beirut, October 3, 2014

Duration: 60 minutes

Last Name: 1 Solution

First Name: Key

Student number: AB XXXXX

No Calculators or other aids are allowed
(cell phones must be turned off during this exam)
No credit will be given if the answers are not justified.
No books, no course notes, no cheat sheets are allowed

For marker's use only	
Question	Mark
1	/7
2	/7
3	/7
4	/4
Total	/25

[7 marks] **Problem 1.** **(the parts of this problem are not related)**

a) **(4 marks)** Use a "geometric method" to find the general form of functions $u(x,y)$, $(x,y) \in \mathbb{R}^2$, solving the partial differential equation

$$(\sin y)u_x + (\sin x)u_y = 0. \qquad (1)$$

Let $V(x,y) = (\sin y, \sin x)$

Then (1) can be written as

$V \cdot \nabla u = 0$ for all (x,y).

But this means that $V \perp \nabla u$.

1

& we already know that ∇u is $\perp$ to the level curves of $u(x,y)$ through point (x,y).

Thus $V(x,y)$ must be tangent to level curves of u.

Letting $y = y(x)$ be a level curve of u,

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we know that: $\frac{dy}{dx}$ = slope of tangent = slope of $V = \frac{\sin x}{\sin y}$

i.e $\frac{dy}{dx} = \frac{\sin x}{\sin y}$

$\Leftrightarrow \sin y \, dy = \sin x \, dx$

$\Leftrightarrow -\cos y = -\cos x + C_1$ (C_1 is a cte.

$\Leftrightarrow \cos x - \cos y = C_1$.

u must be constant on each of those line curves

So $\boxed{u(x,y) = f(\cos x - \cos y)}$

b) **(3 marks)** Classify the following PDEs (where u is the unknown function) into linear, nonlinear, homogeneous or inhomogeneous (a PDE may belong be of more than one type. In such cases, mention all the classes it belongs to):

(i) $u_t = u_{xx} + u - u^2$

This is a nonlinear PDE.

& $u_t - u_{xx}$ ~~=~~ $u + u^2 = 0$ leads to putting the PDE in the class of homogeneous PDEs.

(ii) $|\nabla u(x,y)| = 1$ (where $|\mathbf{v}|$ stands for the magnitude of a vector $\mathbf{v} \in \mathbb{R}^2$).

Nonlinear $\quad |\nabla u| = \sqrt{u_x^2 + u_y^2} = 1$

Nonhomogeneous

(iii) $q_1(x,y)u_x + q_2(x,y)u_y = g(x,y)$ (for some fixed functions q_1, q_2 and g.)

Nonhomogeneous.

But Linear

[7 marks] **Problem 2.**

a) **(1 point)** What is the Fourier Series of the 2π-periodic function defined by

$$\phi(x) := \frac{3}{2} + \cos(4x) + \sin(3x) \text{ over } [-\pi, \pi]? \text{ Explain your answer.}$$

1

$\cos 4x$ & $\sin 3x$ are 2π-periodic (as they are $\frac{2\pi}{4}$ & $\frac{2\pi}{3}$ periodic already).

So $\frac{3}{2} + \cos 4x + \sin 3x$ is a Fourier representation of ϕ. (all $n \geq 5$ terms of a_n & b_n are 0

b) **(2 marks) (not related to a))** Find the Fourier series of the 2π-periodic function f defined by

$$f(x) := x^2 \quad \text{for} \quad x \in [-\pi, \pi).$$

2

f is even over $[-\pi, \pi)$.

So $b_n = 0 \ \forall n$.

$2L = 2\pi$

$\boxed{L = \pi}$

$$a_0 = \frac{1}{L}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\Big|_0^{\pi}\right) = \frac{2}{\cancel{\pi}}\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

$\boxed{\frac{\pi}{L} = 1}$

$\boxed{n \geq 1}$

$$a_n = \frac{2}{L}\int_0^L x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx\,dx}_{dv} \overset{\text{Int by parts}}{=} \frac{2}{\pi}\left(\frac{x^2}{n}\sin nx\Big|_0^{\pi} - 2\int_0^{\pi} \underbrace{\frac{x\sin nx}{n}\,dx}_{dv}\right)$$

$$= \frac{2}{\pi}\left(0 - \frac{2}{n}\left(-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} -\frac{1}{n}\cos nx\,dx\right)\right)$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{2}{n}\times\frac{\cancel{\pi}}{n}(-1)^n \cancel{- \frac{4}{n^2}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi}}\right]$$

$$= \frac{4}{n^2}(-1)^n.$$

$$FS(f(x)) = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2}\cos nx$$

c) **(2 marks)** Discuss the pointwise and uniform convergence of the Fourier series associated to the function f in part b) above.

f & f' are piecewise continuous over $\mathbb{R}$.

so there is pointwise conv. of $FS(f)$ to f:

since f is cont everywhere

$$\frac{f(x^+)+f(x^-)}{2} = f(x) = \left(4\sum \frac{(-1)^n}{n^2}\right) + \frac{\pi^2}{3}.$$

1

for uniform conv:

f & f' are continuous over any $[a,b]$ which lies strictly in $[-L,L]$.

So $FS(f)$ converges uniformly to f over intervals $[a,b] \subsetneq [-L,L]$.

1

d) **(2 marks)** Deduce, from the above part(s), the values of the infinite sums

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

→ as we said in (c) there is ptwise conv.

1

for $x=0$, $\cos nx = \cos 0 = 1 \quad \forall n$.

so $FS(f)(0) = \frac{\pi^2}{3} + 4\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cdot 1$

$$= \frac{f(0^+)+f(0^-)}{2} = \frac{0+0}{2} = 0.$$

Thus $\boxed{\sum_{n\geq 1}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}}$

1

for $\sum \frac{1}{n^2}$, we take $x=\pi$. $\cos n\pi = (-1)^n$.

$$\frac{f(\pi^+)+f(\pi^-)}{2} = \frac{2\pi^2}{2} = \pi^2 = 4\sum_{n\geq 1}\frac{(-1)^{2n}}{n^2} + \frac{\pi^2}{3}.$$

$$\therefore \sum_{n\geq 1}\frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{2\pi^2}{12} = \frac{\pi^2}{6}.$$

[7 marks] **Problem 3.** Let

$$f_n(x) := (\cos x)^n + 10 \text{ for all } x \in [0, \pi/2], \text{ and } g_n(x) = \frac{x}{1+nx^2} \text{ for all } x \in \mathbb{R}.$$

a) **(2 marks)** Determine the pointwise limit f of the sequence $\{f_n\}$.

For $x \neq 0$ & in $(0, \frac{\pi}{2}]$:

$0 \leq \cos x < 1$ so $(\cos x)^n \xrightarrow[n\to\infty]{} 0$ & $f_n(x) \xrightarrow[n\to\infty]{} 10$

for $x = 0$, $(\cos x)^n = 1^n = 1 \xrightarrow[n\to\infty]{} 1$.

So $f_n(0) \longrightarrow 11$.

2

Let $f(x) = \begin{cases} 10 & \text{for } x \neq 0 \\ 11 & \text{for } x = 0 \end{cases}$ $\boxed{f_n \longrightarrow f \text{ ptwise}}$

b) **(2 marks)** Does $\{f_n\}$ converge uniformly to f over $[0, \pi/2]$? Justify your answer.

Note that $\forall n$, $f_n(x)$ is Cont over $[0, \pi/2]$. While f is NOT.

2

Thus, the conv of f_n to f is Just PTwi & NOT Uniform.

c) **(3 marks)** Show that $g_n \longrightarrow 0$ **uniformly** over $\mathbb{R}$.

Indeed: $g_n(x) \xrightarrow[n\to\infty]{} 0$ $\forall x$.

This means $g_n \longrightarrow 0$ ptwise.

Moreover $\max_{x\in\mathbb{R}} \left| \frac{x}{1+nx^2} \right|$ can be computed!

derivative $= \frac{1+nx^2 - 2nx^2}{(\quad)^2} = \frac{1-nx^2}{(\quad)^2} = 0 \iff x = \mp\sqrt{\frac{1}{n}}$

x	$-\infty$		$-\sqrt{1/n}$		$\sqrt{1/n}$		$+\infty$
f_n'		$-$	0	$+$	0	$-$	
		$\searrow$	$\frac{-\sqrt{1/n}}{1+1}$	$\nearrow$	$\frac{\sqrt{1/n}}{1+1}$	$\searrow$	

, Thus $\max_{x\in\mathbb{R}} \left| \frac{x}{1+nx^2} \right| = \frac{1}{2\sqrt{n}} \xrightarrow[n\to\infty]{} 0$.

Thus $g_n \longrightarrow 0$ Uniformly.

Extra space for Problem 3.

[4 marks] **Problem 4.** Let $u(t, x)$ be a function that satisfies the PDE

$$u_t = u_{xx} + u - u^2, \tag{2}$$

and set $u(t, x) = \phi(x - ct)$ where ϕ is a real-valued function of one variable (denote it by $s = x - ct$). Write down the differential equation satisfied by ϕ whenever u satisfies (2) (Do NOT solve an of the differential equations).

$$\left.\begin{aligned} u_t &= -c\phi'(s) \\ u_x &= 1.\phi'(s) \\ u_{xx} &= \phi''(s). \end{aligned}\right\}$$ by the chain rule

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$u = \phi$

Thus (2) ~~is~~ equivalent to.

$$\boxed{-c\phi' = \phi'' + \phi - \phi^2}$$ 1

or $$\boxed{\phi'' + c\phi' + \phi - \phi^2 = 0}$$