MATH 212 HOMEWORK (# 1)
Due Monday September 22, in class

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Problem 1: PDEs solved via ODEs

a) Show that $\phi(s) := \tanh(s) := \frac{e^s - e^{-s}}{e^s + e^{-s}}$ is a solution of the ordinary differential equation

$$-\phi''(s) + 2\left(\phi(s)^2 - 1\right)\phi(s) = 0, \quad s \in \mathbb{R}, \tag{1}$$

such that $\lim\limits_{s\to-\infty} \phi(s) = -1$ and $\lim\limits_{s\to+\infty} \phi(s) = 1$. For simplicity of the presentation presentation, you may omit the s variable in (1) and just write

$$-\phi'' + 2\left(\phi^2 - 1\right)\phi = 0 \text{ in } \mathbb{R}.$$

b) Consider the partial differential equation

$$-\Delta u + 2(u^2 - 1)u = 0 \text{ for } (x, y) \in \mathbb{R}^2, \tag{2}$$

where u is a function of 2 variables ($u = u(x, y)$) and Δu stands for $u_{xx} + u_{yy}$. Let $\mathbf{e} := (\cos\alpha, \sin\alpha)$ be a unit vector ($|\mathbf{e}| = 1$) for some $\alpha \in (0, \pi/2)$. Consider the change of variables $u(x, y) = \phi(x\cos\alpha + y\sin\alpha)$.

1) Find the differential equation that the function ϕ satisfies whenever $u(x, y)$ satisfies (2).

2) Use Part a) above to find a solution u for PDE (2) which satisfies $\lim\limits_{(x,y)\cdot\mathbf{e}\to-\infty} u(x, y) = -1$ and $\lim\limits_{(x,y)\cdot\mathbf{e}\to+\infty} u(x, y) = 1$ (Here, $(x, y) \cdot \mathbf{e}$ stands for the dot product between the vectors (x, y) and $\mathbf{e}$.)

Problem 2: Order 1 PDEs

1) Find a bunch of functions (3 or more) $f(x, y)$ whose level curves are circles centred at the origin. Explain how you design these functions.

2) What are the components of $V(x, y)$ which is the tangent vector, at a point (x, y), to a circle centred at the origin.

3) In this part, we intend to use the previous collected information in solving a PDE. Consider the partial differential equation

$$-y\partial_x u + x\partial_y u = 0 \text{ for all } (x, y) \in \mathbb{R}^2. \tag{3}$$

Use a geometric method to find the general solution of the PDE (3) above. (Give all necessary explanations.)

4) Can you design a PDE whose solution $u(x, y)$ admits elliptical level curves? (Hint: use the algorithm sketched in parts 1,2,3 above in the case of circles...)

5) Consider the parabolas $(P_a)_{a\in\mathbb{R}}:\ y = x^2 + a$ (a is a random constant). Find a vector $V(x, y)$ which is tangent to P_a at point (x, y).

6) Use the previous part to construct a PDE whose solution $u(x, y)$ will have parabolic level curves of the form $(P_a)_{a\in\mathbb{R}}$.

Problem 3: Change of Coordinates Method

Use a suitable change of coordinates to solve (i.e find a general solution $u(t, x)$) the PDE

$$9u_t - 2014u_x = 0. \tag{4}$$