Math 212 , Fall 2007

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Final Exam, 28 January 2008

1. (10 points) Consider the Dirichlet problem in a square,

$$(\mathbf{P}) \qquad \begin{aligned} & u_{xx} + u_{yy} = -F(x,y), \\ & (0 \le x \le \pi,\ 0 \le y \le \pi), \\ & u(0,y) = u(\pi,y) = u(x,0) = u(x,\pi) = 0, \end{aligned}$$

where F is a given continuous function. Find a formula for the solution u of **(P)** in form of a double Fourier series which involves terms of the form $\sin mx \sin ny$, $(m, n \ge 1)$.

Hint : The function $v(x,y) \equiv v_{mn}(x,y) := (m^2+n^2)^{-1} \sin mx \sin ny$ satisfies the equation

$$v_{xx} + v_{yy} = -\sin mx \sin ny.$$

2. Find the Fourier transforms of the following functions:

(a) (10 points) $\qquad f_1(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

(b) (10 points) $\qquad f_2(x) = 2e^{-(x-1)^2/2}.$

3. Consider the following heat problem with dissipation:

$$(\mathbf{Q}) \qquad \begin{aligned} & \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - tu, \quad (x \in \mathbb{R},\ t > 0), \\ & u(x,0) = e^{-|x|}, \quad (x \in \mathbb{R}), \\ & u(x,t) \to 0 \text{ as } |x| \to \infty, \text{ uniformly in } t. \end{aligned}$$

(a) (10 points) Let $\hat{u}$ denote the Fourier transform of u w.r.t. the variable x. Obtain an ODE for $\hat{u}$. Show that

$$\hat{u}(\xi,t) = e^{-\xi^2 t - (t^2/2)} \frac{2}{1+\xi^2}. \tag{1}$$

(b) (5 points) Use formula (1) to show that the solution of (**Q**) is given by

$$u(x,t) = \frac{2}{\pi} e^{-t^2/2} \int_0^{+\infty} \frac{e^{-\xi^2 t} \cos \xi x}{1+\xi^2}\, d\xi. \tag{2}$$

(c) (5 points) Use the well-known property

$$\mathcal{F}^{-1}\left[\hat{f}(\xi)\hat{g}(\xi)\right] = (f * g)(x), \tag{3}$$

($|f|^2, |g|^2$ integrable), to obtain an alternative representation formula for the solution u of **(Q)**.