Math 210

FINAL EXAMINATION, JUNE 4, 2004

1. (10 pts.) A function f is defined by $f(x) = x^3 \sin(\frac{1}{x^2})$, for $x \neq 0$, and $f(0) = 0$. Prove that f is differentiable and uniformly continuous on R.

2. (10 pts.) A sequence a_n is defined by $a_{n+1} = a_n + \frac{1}{a_n^2}$, and $a_1 = 1$. Obtain the asymptotic behavior of a_n, and prove your answer.

3. (10 pts.) Let $f : K \to R$ be a continuous function, where K is a closed and bounded interval. For $a \in R$ put $E = \{x \in K : f(x) \geq a\}$. Prove that E is a compact subset of R.

4. (10 pts.) Let $f : R \to R$ be a differentiable function, and assume that $|f'(x)| \leq A < 1$ for all $x \in R$. A sequence x_n is defined by $x_{n+1} = f(f(x_n))$. Prove that x_n is a convergent sequence irrespective of the value of x_1.

5. (10 pts.) Let f be given by $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$. Prove that f is defined and continuous on $-1 \leq x \leq 1$. Prove also that f is differentiable on $(-1, 1)$ and find its derivative.

6. (10 pts.) Consider the infinite series

$$\sum_{n=1}^{\infty} [nxe^{-nx^2} - (n-1)xe^{-(n-1)x^2}].$$

If S_n is the n^{th} partial sum, compute S_n, $\lim_{n\to\infty} S_n$, and determine whether or not the series converges uniformly in a nbhd $(-\delta, \delta)$ of 0.

7. (10 pts.) Evaluate the following limits and prove your answers:

$$(a) \lim_{n\to\infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx, \quad (b) \lim_{n\to\infty} \int_0^1 \frac{nt^{n+2}}{1+t} dt.$$

8.(10 pts.) State without proof four " deeper" properties of continuous functions.Prove that if f is continuous at a, and $f(a) \neq 0$, then $1/f$ is continuous at a.

9. (10 pts.) Define

$$f(x) = \int_x^{x+1} \sin(t^3) dt.$$

Show that

$$f(x) = \frac{1}{3}\{\frac{\cos(x^3)}{x^2} - \frac{\cos[(x+1)^3]}{(x+1)^2}\} - \frac{2}{9}\int_{x^3}^{(x+1)^3} \frac{\cos u}{u^{5/3}} du,$$

and conclude that $|f(x)| < C/x^2$, where C is a constant; and that $\int_0^{\infty} \sin(t^3) dt$ converges.

10. (10 pts.) If $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, prove that $f'(x) = \sum_{n=1}^{\infty} \frac{-\sin nx}{n}$, for all $x \in (0, 2\pi)$.

11. (10 pts.) Prove that $\int_0^{\infty} \frac{\sin x}{\sqrt{1+x^2}} dx$ is a convergent integral, which is not absolutely convergent.

12. (15 pts.) Let ϕ and S_0 be continuous on $I : a \leq x \leq b$. Define a sequence S_n of functions on I by

$$S_n(x) = \alpha + \int_a^x \phi(t) S_{n-1}(t) dt, n \geq 1, \alpha \text{ fixed in } I.$$

(a) If S_n converges uniformly to a function S on I, prove that S is differentiable and that $S'(x) = \phi(x)S(x)$. (b) Show that $\{S_n\}$ is uniformly convergent on I.

1. (a) Use mathematical induction to prove that $1.2 + 2.3 + \cdots n(n+1)$ $= n(n+1)(n+2)/3$. (6 points)

(b) Show that $n^2 \not\equiv 2 \pmod 6$, $n \in \mathbf{Z}$. State whether your proof is direct or indirect? (6 points)

2. Let $A = \mathbf{R} \times (-\pi, \pi]$. $B = (\mathbf{R} \times \mathbf{R}) \setminus \{(0,0)\}$, and $f : A \to B$ be the function defined by $f(x,y) = (e^x \cos y, e^x \sin y)$.

(a) Graph the sets A and B in a cartesian system. (3 points)

(b) Show that f is a one-to-one correspondence. (6 points)

(c) Find the inverse function of f. (2 points)

3. (a) There are four large groups of people, each with 1000 members. Any two of these groups have 100 members in common. Any three of these groups have 10 members in common. And there is 1 person common to all four groups. All together, how many people are in these groups? (6 points)

(b) There are 10 questions in a Discrete Mathematics final examination. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points? (6 points)

(c) How many different words can be made from "SUCCESS" given that the first and last letters must both be "S"? (6 points)