Chapter 16 : Integration in Vector Fields

16.1 [illegible]

To integrate a continuous function $f(x,y,z)$ over a curve C:

1) Find a smooth parametrization of C:
$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k} \quad a \leq t \leq b$
2) Evaluate the integral as: $\quad \vec{v}(t) = \frac{d\vec{r}}{dt}$
$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$$

If f has the constant value 1, then the integral of f over C gives the length of C

Additivity (line integral has the property that if a curve C is made by joining a finite number of curves $C_1, C_2, \ldots, C_m$ then:
$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_m} f\,ds$$

[illegible] $M = \int_C \delta(x,y,z)\,ds$

16.2 [illegible]

[illegible]

The gradient field of a differentiable function $f(x,y,z)$ is the field of gradient vectors
$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

[illegible]

The work done by a force $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ over a smooth curve $r(t)$ from $t=a$ to $t=b$ is:
$$W = \int_{t=a}^{t=b} \vec{F}\cdot\vec{T}\,ds$$

Work integral $W = \int_{t=a}^{t=b} \vec{F}.\vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}\cdot d\vec{r}$

$$= \int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \int_a^b \left(M\frac{dg}{dt} + N\frac{dh}{dt} + P\frac{dk}{dt}\right)dt$$

because $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$

$$W = \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt$$

$$\boxed{W = \int_a^b M\,dx + N\,dy + P\,dz}$$

To evaluate the work integral along a smooth curve $\vec{r}(t)$:

1- Evaluate $\vec{F}$ on the curve as a function of the parameter t

2- Find $\frac{d\vec{r}}{dt}$

3- Integrate $\vec{F}.\frac{d\vec{r}}{dt}$ from $t=a$ to $t=b$

Work $\Leftrightarrow$ Forget these names!

If $\vec{r}(t)$ is a smooth curve in the domain of a continuo… velocity field $\vec{F}$, the flow along the curve from $t=a$ to $t=$…

Flow $= \int_a^b \vec{F}.\vec{T}\,ds$

We should call everything Work

If C is a smooth closed curve in the domain of a continuous vector field $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ in the plane and if $\vec{n}$ is the outward-pointing unit normal vector on C, the flux of $\vec{F}$ across C is:

Flux of $\vec{F}$ across C : $\int_C \vec{F}.\vec{n}\,d\vec{s}$

$\vec{n} = \vec{T}\times\vec{k}$

or $\vec{n} = \vec{k}\times\vec{T}$ (cross product)

Flux = $\oint_C M\,dy - N\,dx$ the integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \leq t \leq b$, that traces C counterclockwise exactly once.

16.3)

Let $\vec{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B in D the work $\int_A^B \vec{F} \cdot d\vec{r}$ done in moving from A to B is the same over all paths from A to B. Then the integral $\int \vec{F} \cdot d\vec{r}$ is path independent in D and the field $\vec{F}$ is conservative on D

If $\vec{F}$ is a field defined on D and $\vec{F} = \vec{\nabla} f$ for some scalar function f on D, then f is called a potential function for $\vec{F}$

1- Let $\vec{F} = \langle M, N, P \rangle$ be a vector field whose components are continuous throughout an open connected region D in space. Then, there exists a differentiable function f such that:
$\vec{F} = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ if and only if for all points A and B in D the value of $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path joining A to B in D.

2- If the integral is independent of the path from A to B, its value is

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

The following statements are equivalent.

1- $\oint \vec{F} \cdot d\vec{r} = 0$ around every closed loop in D

2- The field $\vec{F}$ is conservative on D

$\vec{F} = \vec{\nabla} f$ on D $\Leftrightarrow$ $\vec{F}$ conservative on D $\Leftrightarrow$ $\oint \vec{F} \cdot d\vec{r} = 0$ over any closed in D

Let $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a whose component functions have continuous first part derivatives. Then, $\vec{F}$ is conservative if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \text{ and } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Once we know that $\vec{F}$ is conservative, we usually wa to find a potential function for $\vec{F}$. This requires solv the equation $\vec{\nabla} f = \vec{F}$ or

$$\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = M\vec{i} + N\vec{j} + P\vec{k} \text{ for } f$$

16.4

The divergence (flux density) of a vector field $\vec{F} = M\vec{i} + N\vec{j}$ at the point (x,y) is

$$\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

The k-component of the curl (circulation density) a vector field $\vec{F} = M\vec{i} + N\vec{j}$ at the point (x,y) is scalar

$$(\operatorname{curl} \vec{F}) \cdot \vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

The outward flux of a field $\vec{F} = M\vec{i} + N\vec{j}$ across a simple closed curve C equals the double integral of div $\vec{F}$ over the region R enclosed by C

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \iint_R \text{div } F \; dx\,dy$$

The counterclockwise circulation of a field $\vec{F} = M\vec{i} + N\vec{j}$ around a simple closed curve C in the plane equals the double integral of $(\text{curl } \vec{F})\cdot\vec{k}$ over the region R enclosed by C

$$\oint_C \vec{F}\cdot\vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

16.5

The area of the surface $f(x,y,z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{p}|}\, dA$$

where $\vec{p}$ is a unit vector normal to R and $\vec{\nabla} f\cdot\vec{p} \neq 0$

If R is the shadow region of a surface S defined by the equation $f(x,y,z) = c$ and g is a continuous function defined at the points of S, then the integral of g over S is the integral

$$\iint_R g(x,y,z)\, \frac{|\vec{\nabla} f|}{|\vec{\nabla} f\cdot\vec{p}|}\, dA$$

If $g = 1$ surface integral = surface area

$$d\sigma = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{p}|} \, dA \Rightarrow \iint_S g \, d\sigma$$

The flux of a three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction of $\vec{n}$ is

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

If S is part of a level surface $g(x, y, z) = c$, then $\vec{n}$ is:

$$\vec{n} = \pm \frac{\vec{\nabla} g}{|\vec{\nabla} g|} \quad \text{depending on direction}$$

with $d\sigma = \dfrac{|\vec{\nabla} g|}{|\vec{\nabla} g \cdot \vec{p}|} \, dA$

$$\text{Flux} = \iint_R \vec{F} \cdot \frac{\pm \vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{p}|} \, dA$$

$$M = \iint_S \delta(x, y, z) \, d\sigma$$

16.6

$$\vec{r}(u, v) = f(u, v) \vec{i} + g(u, v) \vec{j} + h(u, v) \vec{k}$$

$x = f(u, v)$

$y = g(u, v)$ + indicate parameter domain R.

$z = h(u, v)$

A parametrized surface $\vec{r}(u, v) = f(u, v) \vec{i} + g(u, v) \vec{j} + h(u, v) \vec{k}$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the parameter domain.

$$\vec{r_u} = \frac{\delta \vec{r}}{\delta u} = \frac{\delta f}{\delta u}\vec{i} + \frac{\delta g}{\delta u}\vec{j} + \frac{\delta h}{\delta u}\vec{k}$$

$$\vec{r_v} = \frac{\delta \vec{r}}{\delta v} = \frac{\delta f}{\delta v}\vec{i} + \frac{\delta g}{\delta v}\vec{j} + \frac{\delta g}{\delta v}\vec{k}$$

The area of a smooth surface
$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$,
$a \leq u \leq b \;,\; c \leq v \leq d$
is

$$A = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv \qquad A = \iint_S d\sigma$$

If S is a smooth surface defined parametrically as $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$, $a \leq u \leq b$, $c \leq v \leq d$, and $G(x,y,z)$ is a continuous function defined on S, then the integral of G over S is

$$\iint_S G(x,y,z)\,d\sigma = \int_c^d \int_a^b G\big(f(u,v),\, g(u,v),\, h(u,v)\big) \, |\vec{r_u} \times \vec{r_v}| \, du \, dv$$

$$\text{Flux} = \iint \vec{F} \cdot \vec{n} \, d\sigma$$

$$\vec{n} = \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|} \qquad d\sigma = |\vec{r_u} \times \vec{r_v}| \, du \, dv$$

16.7)

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$\vec{\nabla} = \frac{\delta}{\delta x}\vec{i} + \frac{\delta}{\delta y}\vec{j} + \frac{\delta}{\delta z}\vec{k}$$

$$\vec{F} = \langle M, N, P \rangle$$

$$\text{curl } \vec{F} = \left(\frac{\delta P}{\delta y} - \frac{\delta N}{\delta z}\right)\vec{i} + \left(\frac{\delta M}{\delta z} - \frac{\delta P}{\delta x}\right)\vec{j} + \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)\vec{k}$$

[illegible] Theorem

The circulation of a vector field $\vec{F} = \langle M, N, P \rangle$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\vec{n}$ equals the integral of $\vec{\nabla} \times \vec{F} \cdot \vec{n}$ over S

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n} \, d\sigma$$

T[illegible] (conservative)

If $\vec{\nabla} \times \vec{F} = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

16.8 T[illegible]

$\vec{F} = \langle M, N, P \rangle$

$$\text{div}\, \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\delta M}{\delta x} + \frac{\delta N}{\delta y} + \frac{\delta P}{\delta z}$$

The flux of a vector field $\vec{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\vec{n}$ equals the triple integral of $\vec{\nabla} \cdot \vec{F}$ over the region D enclosed by the surface

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F} \, dV$$

$\Rightarrow$

(flux)

Normal form of Green's theorem : $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \vec{\nabla}\cdot\vec{F}\,dA$

Divergence Theorem : $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV$

(work)

Tangential form of Green's Theorem : $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \vec{\nabla}\times\vec{F}\cdot\vec{k}\,dA$

$\left(= \iint \frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\,dA\right)$

Stoke's Theorem : $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,d\sigma$

MATHS 202

SUMMARY CHAPTER 16

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It is not enough to study the ~~theorems~~, you should practice doing a lot of problems

If you have any question contact me @ ERIK VZ on facebook

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Begin to act to Save Environment - I am just exposing to you the small tricks that you can apply easily, giving you contibute to a greener Lebanon