EECE231
Lecture Notes

Topic 11 - Numerical Analysis:

Finding roots in Matlab

Root Finding

Given a function f, we are looking for a value x, s.t.

f(x)=0

(a root of the equation, or a zero of the function f).
The problem is called root finding or zero finding.

- Example of nonlinear equation in one dimension

$$x^2 - 4\sin(x) = 0$$

for which $x = 1.9$ is one approximate solution

Examples

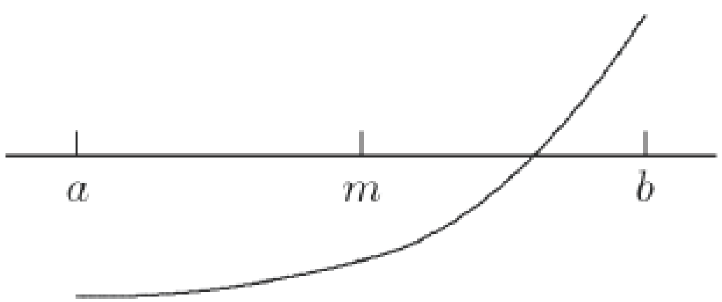
Nonlinear equations can have any number of solutions

- $\exp(x) + 1 = 0$ has no solution
- $\exp(-x) - x = 0$ has one solution
- $x^2 - 4\sin(x) = 0$ has two solutions
- $x^3 + 6x^2 + 11x - 6 = 0$ has three solutions
- $\sin(x) = 0$ has infinitely many solutions

Bisection method

Bisection method begins with initial bracket and repeatedly halves its length until solution has been isolated as accurately as desired

while $((b-a) > tol)$ **do**
    $m = a + (b-a)/2$
    if $\text{sign}(f(a)) = \text{sign}(f(m))$ **then**
        $a = m$
    else
        $b = m$
    end
end



Example: bisection method

$$f(x) = x^2 - 4\sin(x) = 0$$

a	$f(a)$	b	$f(b)$
1.000000	−2.365884	3.000000	8.435520
1.000000	−2.365884	2.000000	0.362810
1.500000	−1.739980	2.000000	0.362810
1.750000	−0.873444	2.000000	0.362810
1.875000	−0.300718	2.000000	0.362810
1.875000	−0.300718	1.937500	0.019849
1.906250	−0.143255	1.937500	0.019849
1.921875	−0.062406	1.937500	0.019849
1.929688	−0.021454	1.937500	0.019849
1.933594	−0.000846	1.937500	0.019849
1.933594	−0.000846	1.935547	0.009491
1.933594	−0.000846	1.934570	0.004320
1.933594	−0.000846	1.934082	0.001736

Bisection method

- Bisection method makes no use of magnitudes of function values, only their signs
- Bisection is certain to converge, but does so slowly
- At each iteration, length of interval containing solution reduced by half, convergence rate is *linear*, with $r = 1$ and $C = 0.5$
- One bit of accuracy is gained in approximate solution for each iteration of bisection
- Given starting interval $[a, b]$, length of interval after k iterations is $(b - a)/2^k$, so achieving error tolerance of tol requires

$$\left\lceil \log_2 \left(\frac{b-a}{tol} \right) \right\rceil$$

iterations, regardless of function f involved

Newton's method

- Truncated Taylor series

$$f(x+h) \approx f(x) + f'(x)h$$

  is linear function of h approximating f near x

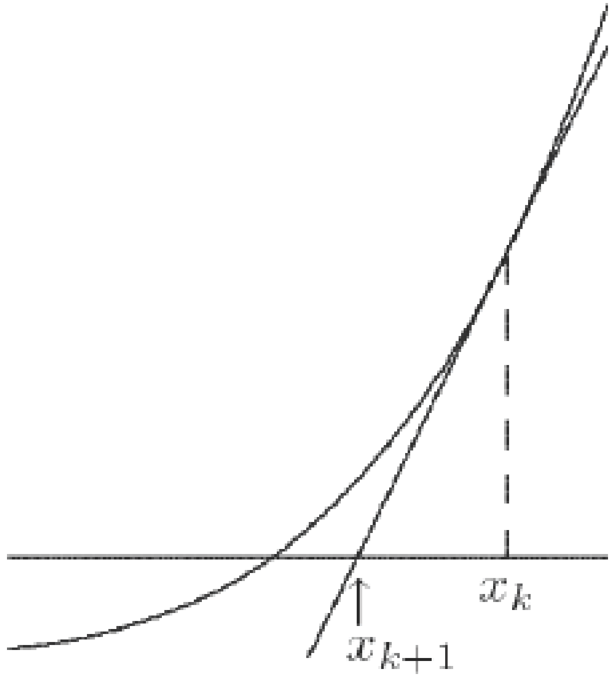
- Replace nonlinear function f by this linear function, whose zero is $h = -f(x)/f'(x)$

- Zeros of original function and linear approximation are not identical, so repeat process, giving *Newton's method*

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

Newton's method

Newton's method approximates nonlinear function f near x_k by *tangent line* at $f(x_k)$



Example: Newton's method

- Use Newton's method to find root of

$$f(x) = x^2 - 4\sin(x) = 0$$

- Derivative is

$$f'(x) = 2x - 4\cos(x)$$

so iteration scheme is

$$x_{k+1} = x_k - \frac{x_k^2 - 4\sin(x_k)}{2x_k - 4\cos(x_k)}$$

- Taking $x_0 = 3$ as starting value, we obtain

x	$f(x)$	$f'(x)$	h
3.000000	8.435520	9.959970	−0.846942
2.153058	1.294772	6.505771	−0.199019
1.954039	0.108438	5.403795	−0.020067
1.933972	0.001152	5.288919	−0.000218
1.933754	0.000000	5.287670	0.000000

Convergence of Newton's method

- Newton's method transforms nonlinear equation $f(x) = 0$ into fixed-point problem $x = g(x)$, where

$$g(x) = x - f(x)/f'(x)$$

  and hence

$$g'(x) = f(x)f''(x)/(f'(x))^2$$

- If x^* is simple root (i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$), then $g'(x^*) = 0$
- Convergence rate of Newton's method for simple root is therefore *quadratic* ($r = 2$)
- But iterations must start close enough to root to converge

Secant method

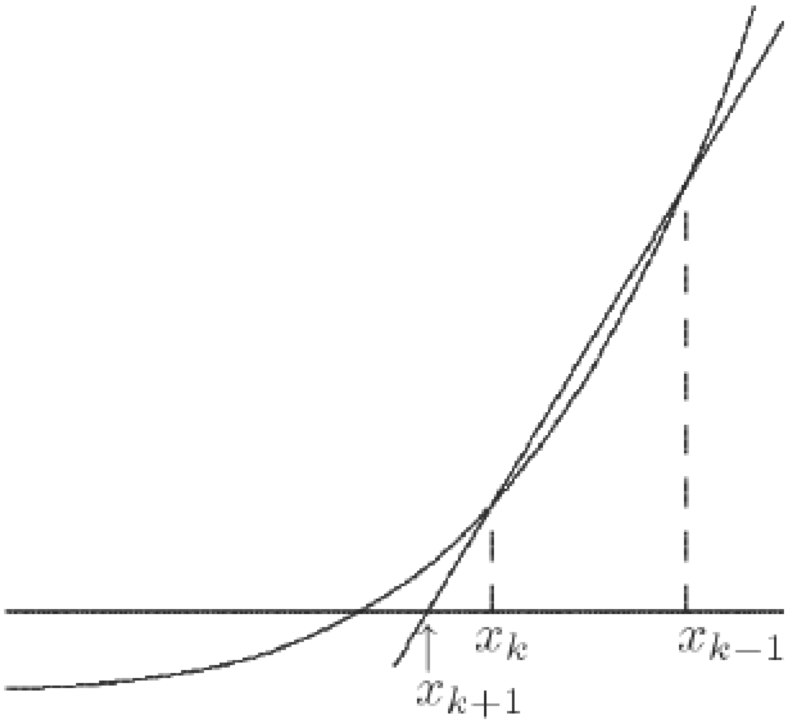
- For each iteration, Newton's method requires evaluation of both function and its derivative, which may be inconvenient or expensive
- In *secant method*, derivative is approximated by finite difference using two successive iterates, so iteration becomes

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

- Convergence rate of secant method is normally *superlinear*, with $r \approx 1.618$

Secant method

Secant method approximates nonlinear function f by secant line through previous two iterates



Example: secant method

- Use secant method to find root of

$$f(x) = x^2 - 4\sin(x) = 0$$

- Taking $x_0 = 1$ and $x_1 = 3$ as starting guesses, we obtain

x	$f(x)$	h
1.000000	−2.365884	
3.000000	8.435520	−1.561930
1.438070	−1.896774	0.286735
1.724805	−0.977706	0.305029
2.029833	0.534305	−0.107789
1.922044	−0.061523	0.011130
1.933174	−0.003064	0.000583
1.933757	0.000019	−0.000004
1.933754	0.000000	0.000000