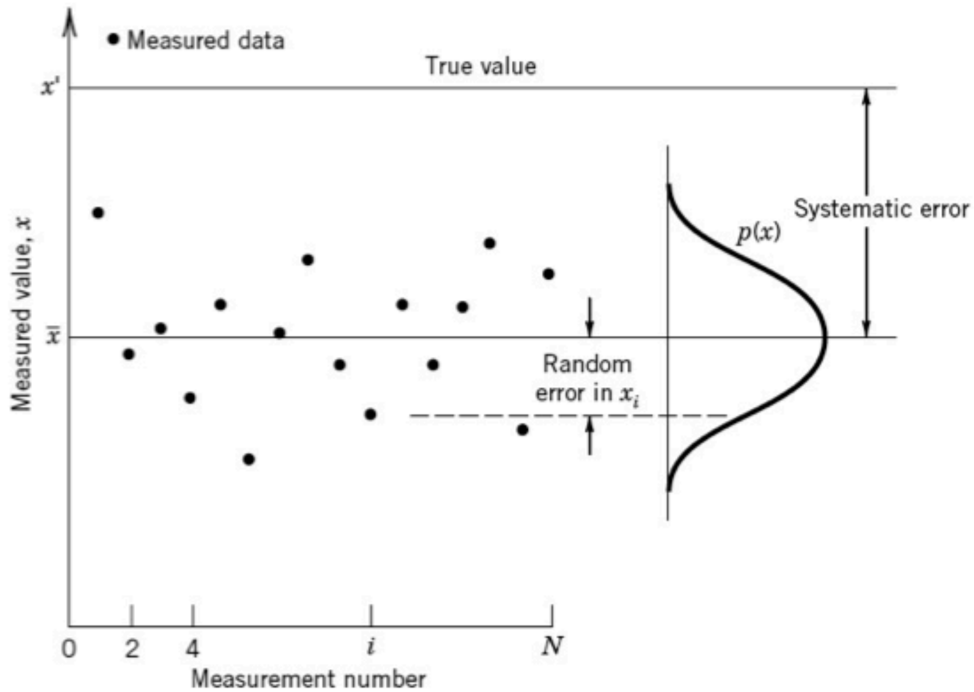
Instrumentation & Measurements MECH 430

Chapter 5 Uncertainty Analysis



Summary of Finite Statistics

Table 4.7 Summary Table for a Sample of N Data Points

Sample mean	$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$
Sample standard deviation	$S_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$
Standard deviation of the means[1]	$S_{\bar{x}} = \frac{S_x}{\sqrt{N}}$
Precision interval for a single data point, x_i	$\pm t_{\nu,P} S_x$ (P%)
Confidence interval[2,3] for a mean value, $\bar{x}$	$\pm t_{\nu,P} S_{\bar{x}}$ (P%)
Confidence interval[2,4] for curve fit, $y = f(x)$	$\pm t_{\nu,P} \frac{S_{yx}}{\sqrt{N}}$ (P%)

[1]Measure of standard random uncertainty in $\bar{x}$.
[2]In the absence of systematic errors.
[3]Measure of random uncertainty in $\bar{x}$.
[4]Measure of random uncertainty in curve fit.

5.1 Introduction

- Uncertainty analysis provides a methodical approach to estimating the accuracy of results.
- In other words we can estimate the "±what" in a planned test.
- Errors causes a difference from the true value, but since we don't know the true value and hence we estimate a range of probable error. This estimate is called uncertainty.
- Uncertainty is a property of the result.

5.1 Introduction

- The uncertainty describes an interval about the measured value within which we suspect that the true value must fall with a stated probability.

$$x' = \bar{x} \pm u_x \quad (P\%)$$

- What we want to do is quantify, on average, how closely the measured value agrees with some known true value and base our interpretation of subsequent measured results on that information.
- Uncertainty quantifies the quality of the result.

5.2 Measurement Errors

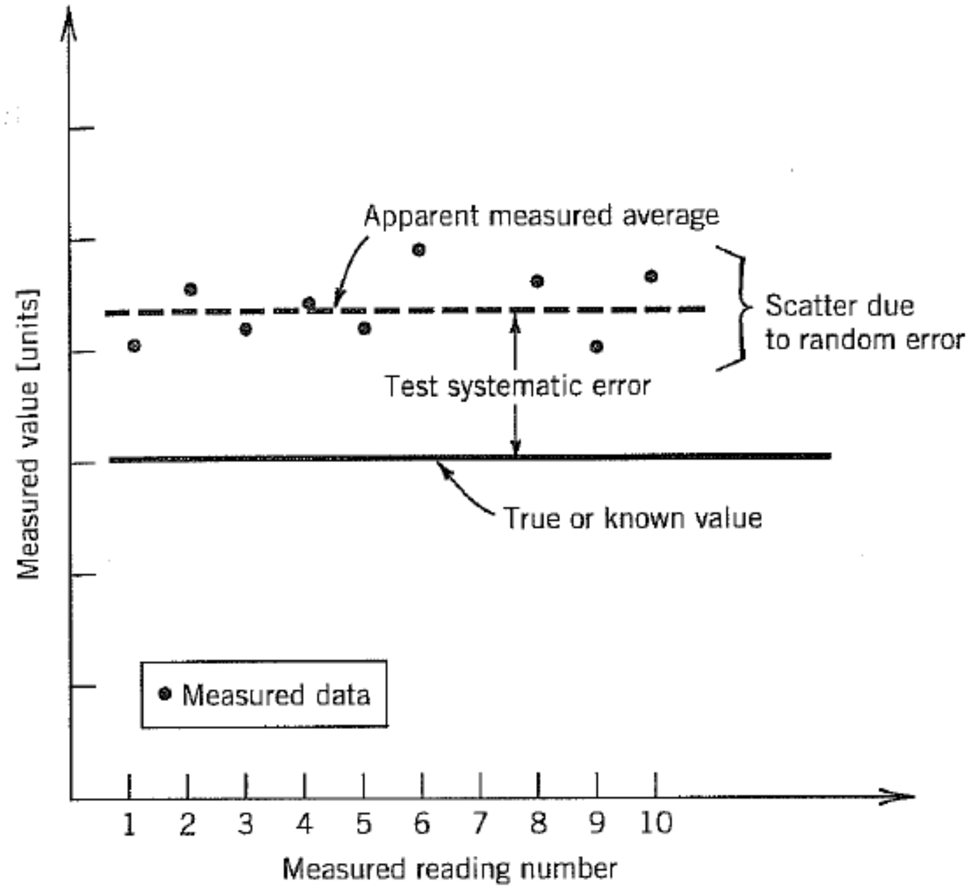
- An error is the difference between a measured value and the true value. An error is a property of a measurement.
- Errors are introduced from various elements:
  - The instrument,
  - The data set finite statistics,
  - The approach used.
- Given that we can not know the true value, we can estimate a range of probable error.
- We call this estimate the *uncertainty*.
- *Uncertainty analysis* is the process of identifying, quantifying, and combining these errors.

5.2 Measurement Errors

- An error is said to be systematic when in the course of measuring the same value of a given quantity under the same conditions, it remains constant in absolute value and sign, or varies according to a definite law when measurement conditions change.
- Random errors are those that remain after eliminating the causes of systematic errors. They bring about a distribution of measured values about the sample mean.

Systematic & Random Errors

- The systematic errors will shift the sample mean from the true mean of the measured variable by a fixed amount.
- The random errors bring about a distribution of measured values about the sample mean.



Systematic Errors

- Systematic errors yield what is known as measurement bias.
- Systematic errors remains constant in repeated measurements under fixed operating conditions.
- Being a fixed value, the systematic error cannot be directly discerned by statistical means alone.
- Sometimes it is even difficult to recognize its presence.
- Systematic errors are estimated by comparison.

Systematic Errors

- The presence (not correction at this point) of systematic errors can therefore be discovered by:
  - Measuring the same quantity with 2 different devices,
  - Using two different methods,
  - Using two different operators,
  - Changing the measurement conditions and observing their influence on results (statistical analysis).
- The correction of systematic error can by achieved by calibrating to a known standard.

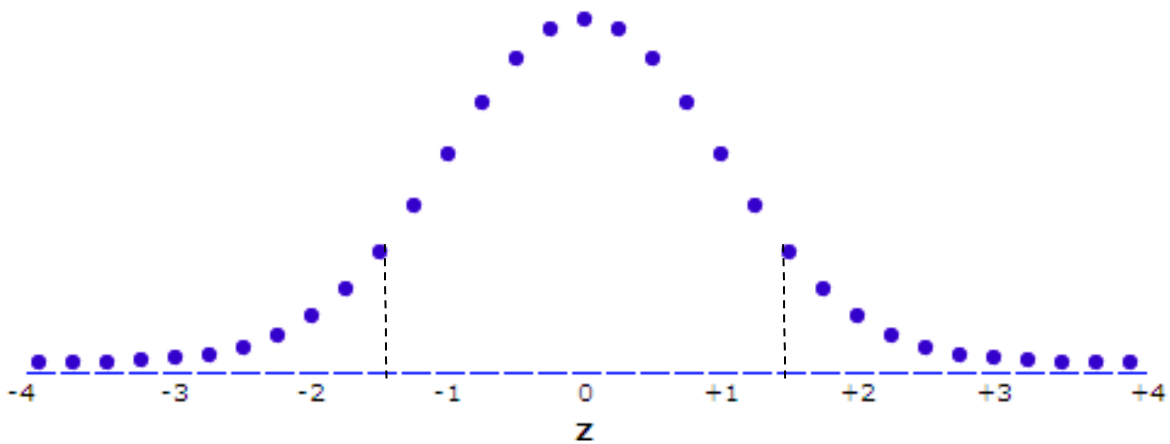
Random Errors

- Random error, also called precision error, is affected by the repeatability and resolution of the measurement system components.

- Because of the varying nature of the random errors, exact values cannot be given, but probable estimates of the error can be made through statistical analyses.

Random Errors

- Random errors are characterized by the following:
  - Positive and negative random errors with the same absolute values have the same occurrence probability.
  - The probability of occurrence of a random error decreases as its absolute value increases.



Random Errors

- In the absence of systematic errors, the best estimate of the actual value of a measurand is obtained by taking n measurements and averaging the results:

$$\hat{x}_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- When n is infinite, (1.3) yields the true value for x.
- However, when n is finite, each experiment that yields a group of x's gives a different average.
- The random uncertainty at a defined confidence level is defined by the interval $\pm t_{v,P} S_{\bar{x}}$ $\quad S_{\bar{x}} = \frac{S_x}{\sqrt{N}}$

5.3 Design Stage Uncertainty

- Analysis done prior to measurement.
- It is useful for selecting instruments, measurement techniques, and estimating the likely uncertainty to exist in the measured data.
- Data at this level is obtained from catalogues.
- It is probably difficult in this stage to distinguish between systematic and random errors.
- The goal of the design stage is to estimate the magnitude of uncertainty in the measured value that would result from a measurement.

5.3 Design Stage Uncertainty

- The *zero-order uncertainty* (u_0) *of an instrument* reflects our ability to resolve the information provided by the instrument.
- At zero-order uncertainty we assume the variation expected in the measured values will be that of the instrument resolution alone and all other aspects of the measurement are perfectly controlled.
- u_0 is an estimate of the expected random uncertainty caused by the data scatter due to reading the instrument.

Zero-order Uncertainty

- As a rule of thumb, assign a numerical value to u_0 of one half of the instrument resolution with a probability of 95%.

$$u_0 = \pm \frac{1}{2}\text{resolution (95\%)}$$

- At 95%, we assume that only one measured value in 20 will have a value outside of u_0.

Instrument Uncertainty

- The second piece of information from a manufacturer is the ***instrument uncertainty*** u_c
- Essentially, u_c is an estimate of the systematic error for the instrument.
- Sometimes the instrument error will be stated in parts, each part due to some contributing factor.
- We combine these errors via the root-sum-squares (RSS) method.

Root Sum Square (RSS) method

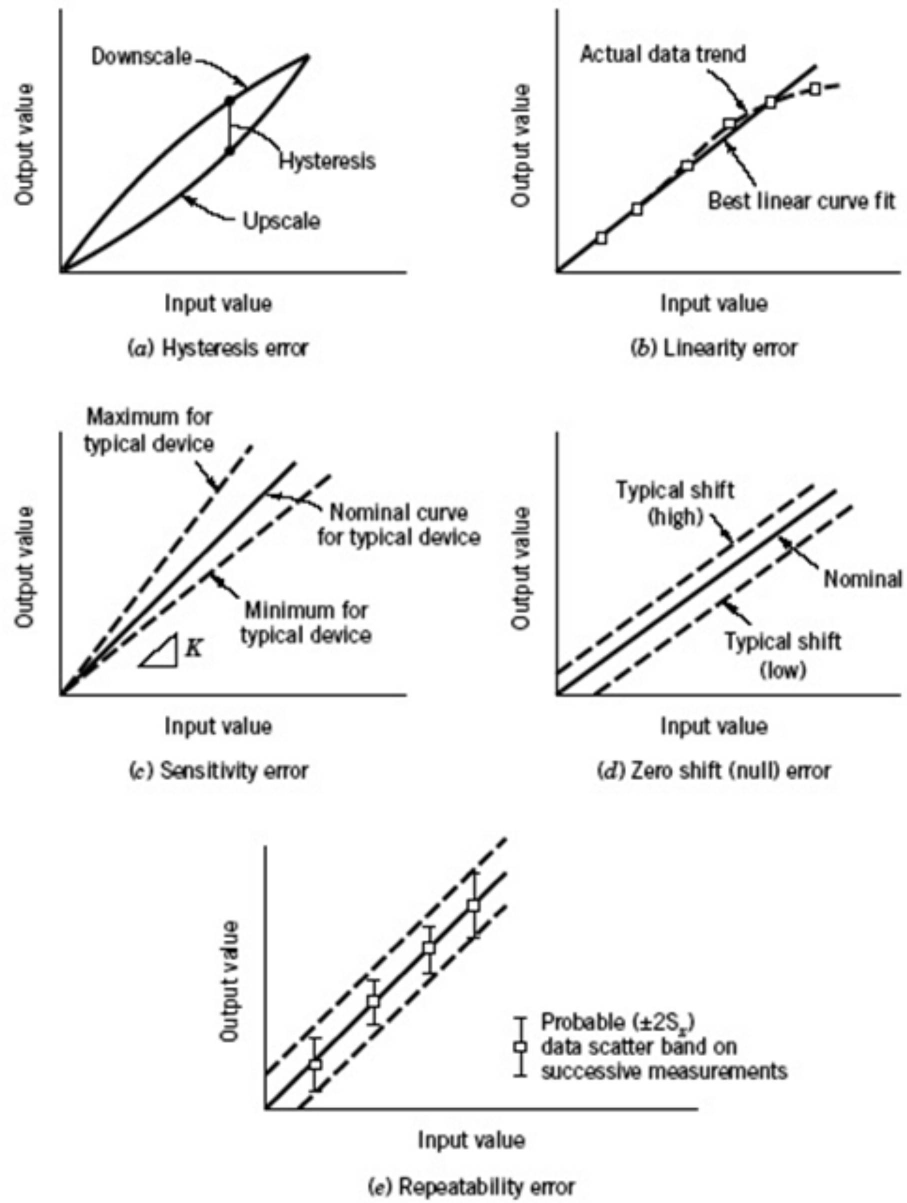
- We will refer to each individual error as an ***elemental error***.
- For example the sensitivity error and the linearity error of a transducer are two elemental errors.
- Consider a measurement of x subject to k elemental errors $e_1, e_2, \ldots e_n$:

$$u_x = \pm\sqrt{e_1^2, e_2^2, \cdots, e_K^2}$$

$$= \pm\sqrt{\sum_{k=1}^{K} e_k^2}$$

Different type of instrument errors:

- Hysteresis
- Linearity
- Sensitivity
- Zero Shift
- Repeatability



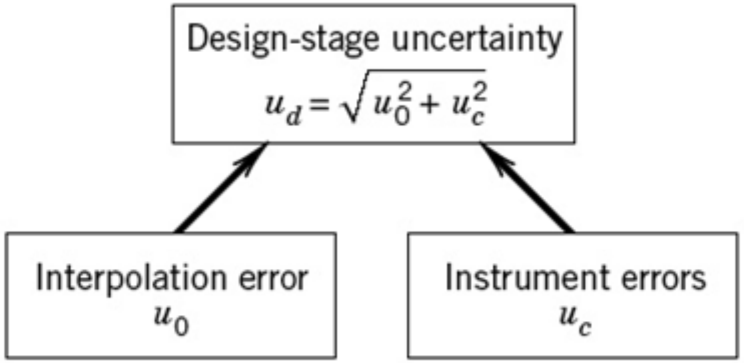
RSS method

- In calculating RSS be careful to be consistent with the units.
- Furthermore, each error should be estimated at the same probability level. A general rule is to use 95%.
- If no probability level is provided on the datasheet, a 95% level can be assumed.

5.3 Design Stage Uncertainty

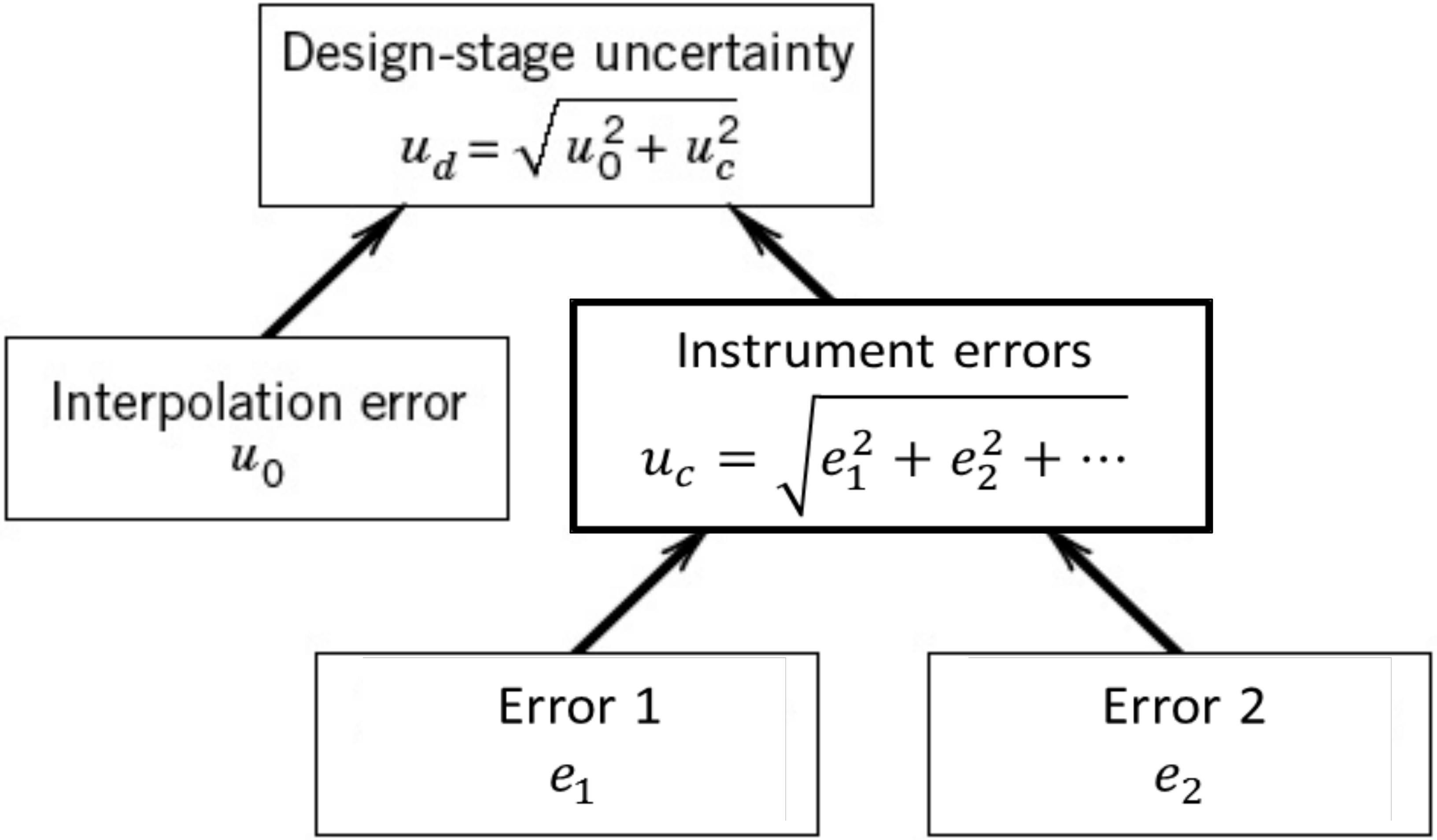
- The design stage uncertainty is then obtained by combining the instrument uncertainty with the zero-order uncertainty:

$$u_d = \sqrt{u_0^2 + u_c^2}$$



- The design stage uncertainty is used as a <u>guide</u> for selecting equipment and is NOT <u>the final estimate</u> of the total uncertainty in a measurement system.
- If <u>additional information</u> is known at this stage then <u>should be included</u> in the above equation.

5.3 Design Stage Uncertainty



Example 5.1

- Given: a force measuring instrument is described by the catalogue data below.
- Estimate: the uncertainties attributable to this instrument and the instrument design stage uncertainty.

Resolution	0.25N
Range	0 to 100N
Linearity	within 0.2 N over range
Hysteresis	within 0.3N over range

Example 5.1

- Solution: first find elemental errors of linearity and hysteresis

$$u_{linearity} = 0.2N$$

$$u_{hysteresis} = 0.3N$$

- Next combine using RSS to find u_c:

$$u_c = \pm\sqrt{0.2^2 + 0.3^2} = \pm 0.36N \quad (95\%)$$

- The instrument resolution is 0.25N so $u_0 = \pm 0.125\text{N}$
- Finally the design-stage uncertainty is:

$$u_d = \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{0.36^2 + 0.125^2} = \pm 0.38N \quad (95\%)$$

Example 5.2

- Given: a voltmeter is to be used to measure the output of a pressure transducer.
- **<u>Pressure transducer</u>** (3psi expected nominal pressure)
  - Range: ±5psi
  - Sensitivity: 1V/psi
  - Input power: 10VDC ±1%
  - Output: ±5V
  - Linearity: within 2.5mV/psi over range
  - Sensitivity: within 2mV/psi over range
  - Resolution: negligible

Example 5.2

- **<u>Voltmeter</u>**
  - Resolution: 10μV
  - Accuracy: within 0.001% reading
- Find:
  - u_c for each device
  - u_d for the measurement

Example 5.2

- Solution:

The design stage uncertainty for the voltmeter is:

$$(u_0)_E = \pm 5\,\mu V \quad (95\%)$$

$$(u_d)_E = \pm\sqrt{(u_0)_E^2 + (u_c)_E^2} = \pm 30.4\,\mu V \quad (95\%)$$

$$(u_c)_E = \pm(3V \times 0.00001) = \pm 30\,\mu V \quad (95\%)$$

Example 5.2

- Solution:

The design stage uncertainty for the pressure transducer is:

$$\left(u_0\right)_P = 0$$

$$\left(u_d\right)_E = \pm\sqrt{\left(u_0\right)_E^2 + \left(u_c\right)_E^2} = \pm 9.61mV \quad (95\%)$$

$$\left(u_c\right)_P = \pm\sqrt{\left(2.5mV/psi \times 3psi\right)^2 + \left(2mV/psi \times 3psi\right)^2} = \pm 9.61mV \quad (95\%)$$

Example 5.2

- Solution:

Finally, the design stage uncertainty for the entire system is:

$$u_d = \pm\sqrt{(u_d)_E^2 + (u_d)_P^2} = \pm\sqrt{(0.030mV)^2 + (9.61mV)^2} = \pm 9.61mV \quad (95\%)$$

Using the sensitivity of 1V/psi, the uncertainty in psi is:

$$u_d = \pm 0.0096\, psi \quad (95\%)$$

5.4 Error Sources

- Design-stage uncertainty does not address all possible errors.
- If we consider the measurement process to consist of 3 steps, and each has an uncertainty associated with it:
  - calibration error,
  - data-acquisition error,
  - data reduction error.

5.4 Error Sources

Calibration errors

Table 5.1 Calibration Error Source Group

Element	Error Source[a]
1	Primary to interlab standard
2	Interlab to transfer standard
3	Transfer to lab standard
4	Lab standard to measurement system
5	Calibration and technique
etc.	

[a] Systematic error and/or random error in each element.

Data reduction errors

Table 5.3 Data-Reduction Error Source Group

Element	Error Source[a]
1	Calibration curve fit
2	Truncation error
etc.	

[a] Systematic error and/or random error in each element.

DAQ errors

Table 5.2 Data-Acquisition Error Source Group

Element	Error Source[a]
1	Measurement system operating conditions
2	Sensor-transducer stage (instrument error)
3	Signal conditioning stage (instrument error)
4	Output stage (instrument error)
5	Process operating conditions
6	Sensor installation effects
7	Environmental effects
8	Spatial variation error
9	Temporal variation error
etc.	

[a] Systematic error and/or random error in each element.
Note: A total input-to-output measurement system calibration will combine elements 2, 3, 4, and possibly 1 within this error source group.

5.6 Error Propagation

- Very often results are obtained through a functional relationship with measured values.
- For example the measurement of flow rate using time and volume. $Q = f(t, V)$
- How do uncertainties in either quantity contribute to uncertainty in flow rate?
- Is the uncertainty in Q more sensitive to uncertainty in t or v?
- How are errors propagated to a calculated result?

5.6 Error Propagation

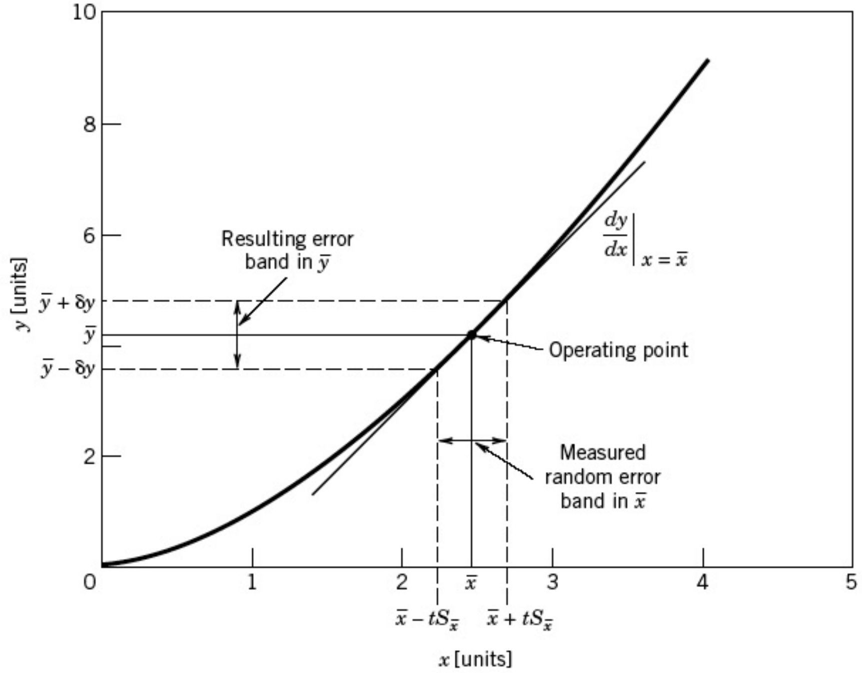


Figure 5.3. Relationship between a measured variable and a resultant calculated using the value of that variable.

- The true value of y falls within the interval

$$\bar{y} \pm \delta y = f(x \pm tS_{\bar{x}})$$

5.6 Error Propagation

$$\bar{y} \pm \delta y = f(x \pm tS_{\bar{x}})$$

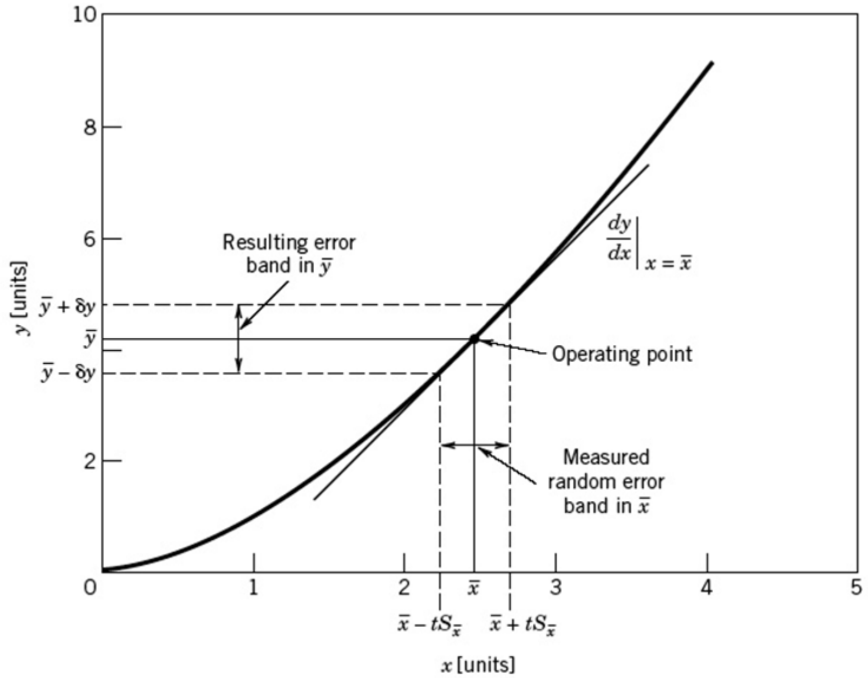
- Expanding using a Taylor series:

$$\bar{y} \pm \delta y = f(\bar{x}) \pm \left[\left(\frac{dy}{dx}\right)_{x=\bar{x}} tS_{\bar{x}} + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{x=\bar{x}} \left(tS_{\bar{x}}\right)^2 + \cdots \right]$$

- If $tS_{\bar{x}}$ is small and if we neglect higher order terms:

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=\bar{x}} tS_{\bar{x}}$$

5.6 Error Propagation



- From above the uncertainty for the dependent variable is related to the uncertainty in the independent variable by:

$$u_y = \left(\frac{dy}{dx}\right)_{x=\bar{x}} u_x$$

Multiple Variables

- For multiple variables with $R = f_1(x_1, x_2, \ldots, x_L)$ the best estimate for the true mean R' is:

$$\bar{R} = f_1(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_L)$$

$$R' = \bar{R} \pm u_R \quad (\text{P\%})$$

$$u_R = \pm\left[\sum_{i=1}^{L}\left(\theta_i u_{\bar{x}_i}\right)^2\right]^{1/2} \quad (\text{P\%})$$

$$\theta_i = \left.\frac{\partial R}{\partial x_i}\right|_{x=\bar{x}} \quad i = 1,2,\ldots,L$$

Example 5.3

- Given:
  - A displacement sensor has a calibration curve $y = KE$,
  - $E = 5\text{V}$,
  - K = 10.10 mm/V
  - $u_K = \pm 0.10$ mm/V (95% confidence)
  - $u_E = \pm 0.01$ V (95% confidence)
- Find:
  - The uncertainty in displacement

Example 5.3

- Solution

$$\bar{y} = f(\bar{E}, \bar{K}) \qquad u_y = f(u_E, u_K)$$

- The uncertainty is found as:

$$\theta_E = \frac{\partial y}{\partial E} = K$$

$$\theta_K = \frac{\partial y}{\partial K} = E$$

$$u_y = \pm\left[\left(\theta_E u_E\right)^2 + \left(\theta_K u_K\right)^2\right]^{1/2}$$

- Therefore:

$$u_y = \pm\left[\left(K(0.01)\right)^2 + \left(E(0.1)\right)^2\right]^{1/2} = \pm 0.51mm \quad (95\%)$$

Sequential Perturbation

- It might be difficult to find the partial derivatives in the previous method.
- We then resort to something known as *sequential perturbation*.
- This method is based on using a finite difference method to approximate the derivatives.

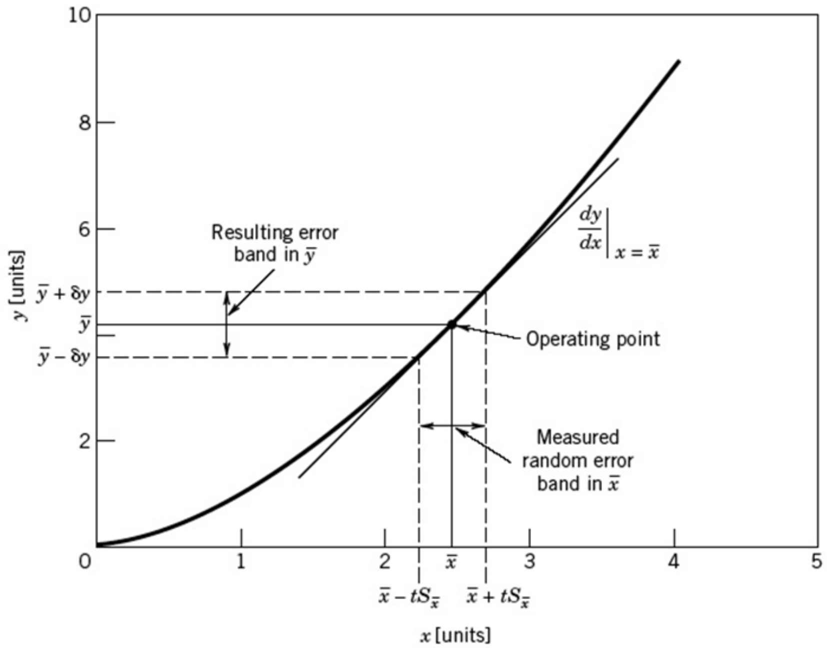
Sequential Perturbation

- Start by finding an operating point $R_o = f(x_1, x_2, \ldots, x_L)$
- Next, increase the independent variables by their uncertainties to find R_i^+

$$R_1^+ = f(x_1 + u_{x_1}, x_2, \ldots, x_L),$$

$$R_2^+ = f(x_1, x_2 + u_{x_2}, \ldots, x_L), \ldots$$

$$R_L^+ = f(x_1, x_2, \ldots, x_L + u_{x_L})$$



- Do the same to find R_i^-
- Calculate the differences: $\delta R_i^+ = R_i^+ - R_o$

$$\delta R_i^- = R_i^- - R_o$$

Sequential Perturbation

- Finally find the approximate uncertainty contribution of each variable.

$$\delta R_i = \frac{\delta R_i^{+} - \delta R_i^{-}}{2} \approx \theta_i u_i$$

- The uncertainty is then calculated as:

$$u_R = \pm \left[\sum_{i=1}^{L} \left(\delta R_i \right)^2 \right]^{1/2} \quad (\mathrm{P}\%)$$

Example 5.3 (continued)

- Alternatively we can use sequential perturbation with $R_o = 50.5$ mm

i	x_i	R_i^+	R_i^-	δR_i^+	δR_i^-	δR_i
1	E	50.60	50.40	0.1	-0.1	0.1
2	K	51.00	50.00	0.5	-0.5	0.5

- The uncertainty is found as:

$$u_y = \pm\left[(0.10)^2 + (0.5)^2\right]^{1/2} = \pm 0.51 mm$$

- Therefore the displacement is:

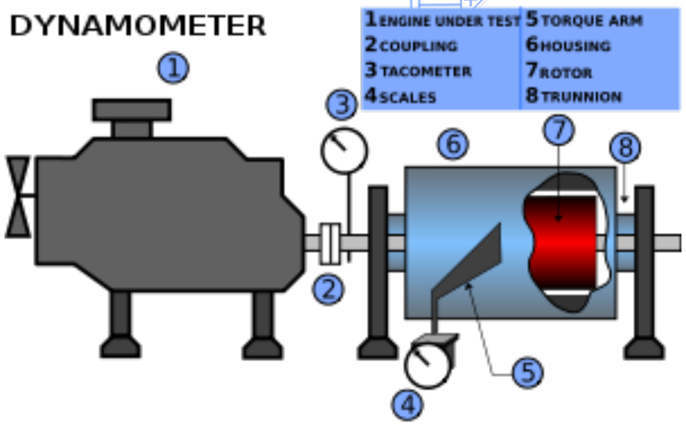
$$y' = 50.50 \pm 0.51 mm \quad (95\%)$$

Principle of equal effects

- If you have the total uncertainty and want to find each individual uncertainty, we can use the ***principal of equal effects.***
- This principal states that each source of uncertainty Δu_i contributes an equal amount to the total uncertainty ΔR.

$$\Delta u_i = \frac{\Delta R}{n\left(\partial R / \partial u_i\right)} \qquad (1)$$

Example



- A dynamometer is used to measure the average power output on a shaft.

$$P = \frac{2\pi}{550\times 12}\frac{RFL}{t}$$

- P: Average power (hp)
- $\frac{2\pi}{550\times 12}$: K
- R: Number of revolutions of the shaft during t seconds
- F: Force at the end of the torque arm (lbf)
- L: Length of torque arm (ft)
- t: Time of run (s)

Example

- Given:
  - For a given run we have the following measurements with their associated errors
  - Find the overall uncertainty:

$$R = 1202 \pm 1 \text{ rev}$$

$$F = 10.12 \pm 0.04 \text{ lbf}$$

$$L = 15.63 \pm 0.05 \text{ in}$$

$$t = 60.00 \pm 0.55 \text{ s}$$

Example

- Solution:
  - Find the partial differentials:

$$\frac{\partial P}{\partial F} = \frac{KLR}{t} = 0.298 \text{ hp/lbf}$$

$$\frac{\partial P}{\partial R} = \frac{KFL}{t} = 0.0251 \text{ hp/rev}$$

$$\frac{\partial P}{\partial L} = \frac{KFR}{t} = 0.193 \text{ hp/in}$$

$$\frac{\partial P}{\partial t} = \frac{KFLR}{t^2} = -0.051 \text{ hp/s}$$

$$F = 10.12 \pm 0.04 \text{ lbf}$$

$$R = 1202 \pm 1 \text{ rev}$$

$$L = 15.63 \pm 0.05 \text{ in}$$

$$t = 60.00 \pm 0.55 \text{ s}$$

Example

- The overall uncertainty is found as:

$$\Delta R_{RSS} = \sqrt{(0.04 \times 0.298)^2 + (1 \times 0.00251)^2 + (0.05 \times 0.193)^2 + (0.55 \times 0.05)^2}$$

$$\Delta R_{RSS} = 0.029 \text{ hp}$$

- If we wish to measure P to within 0.5% uncertainty, what are the errors associated with each individual measurement?

- The power is calculated as:

$$P = \frac{KFLR}{t} = 3.016 \text{ hp}$$

Example

- Using the principle of equal effects:

$$\Delta F = \frac{\Delta P}{n\left(\partial P / \partial F\right)} = \frac{0.005 \times 3.016}{4\left(0.298\right)} = 0.013 \text{ lbf}$$

$$\Delta R = \frac{\Delta P}{n\left(\partial P / \partial R\right)} = \frac{0.005 \times 3.016}{4\left(0.00251\right)} = 1.5 \text{ rev}$$

$$\Delta L = \frac{\Delta P}{n\left(\partial P / \partial F\right)} = \frac{0.005 \times 3.016}{4\left(0.193\right)} = 0.019 \text{ in}$$

$$\Delta t = \frac{\Delta P}{n\left(\partial P / \partial F\right)} = \frac{0.005 \times 3.016}{4\left(0.05\right)} = 0.075 \text{ s}$$

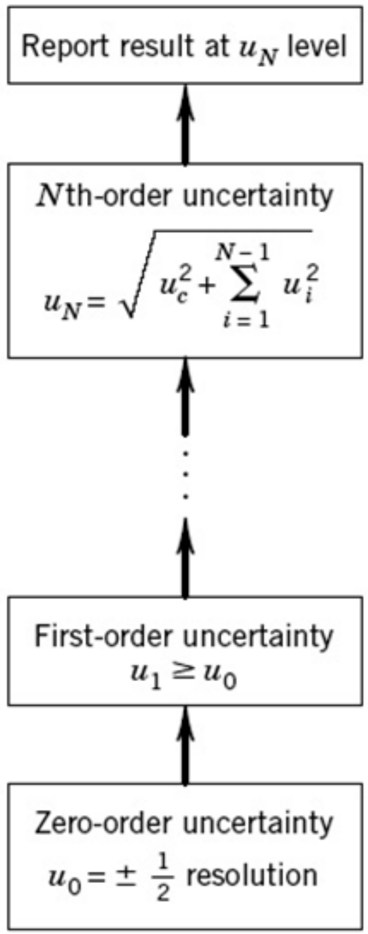
5.7 Advanced Stage Uncertainty

- Higher order uncertainty considers the controllability of test operating conditions and the variability of all measured variables.
- For example, at a first order level, <u>the effect of time</u> as an extraneous variable could be considered.

$$u_N = \left[\sqrt{u_c^2 + \left(\sum_{i=1}^{N-1} u_i^2\right)} \right]^{1/2} \quad (P\%)$$

$$u_0 = \pm \frac{1}{2} \quad \text{resolution}$$

$$u_1 = \pm t_{(J-1),95} S_{\overline{T}}$$



Example 5-4

- Examine a dial oven thermometer and assess the zero and first order uncertainties if we have *J* measurements

$$u_0 = \pm\frac{1}{2} \quad \text{resolution}$$

$$u_1 = \pm t_{(J-1),95} S_{\bar{T}}$$

Example

- A stopwatch is to be used to estimate the time between the start and end of an event. Event duration might range from several seconds to 10 minutes. Estimate the probable uncertainty in a time estimate using a stopwatch that claims an accuracy of 1 minute per month and a resolution of 0.01 seconds.

Example

- **Zero** stage uncertainty

$$u_0 = \pm 0.01/2s$$

$$u_c = \pm \frac{60s}{1(30)(24)(60)(60)s} T$$

$$T = 10s$$

$$\Rightarrow u_d = \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{0.005^2 + \left(10\frac{60}{1(30)(24)(60)(60)}\right)^2} = \pm 0.005s$$

$$T = 10\,\text{min} = 600s$$

$$\Rightarrow u_d = \pm\sqrt{u_0^2 + u_c^2} = \pm\sqrt{0.005^2 + \left(10(600)\frac{60}{1(30)(24)(60)(60)}\right)^2} = \pm 0.015s$$

Example

- Now assume that the operator takes some time to detect the start and end of the event. Assume this uncertainty is quantified as 150ms with 95% probability.

$$u_0 = \pm 0.01/2s$$

$$u_c = \pm \frac{60s}{1(30)(24)(60)(60)s} T$$

$$u_1 = \pm 0.150s$$

$$t = 10s$$

$$\Rightarrow u_N = \pm\sqrt{u_1^2 + u_c^2} = \pm\sqrt{0.15^2 + \left(10\frac{60}{1(30)(24)(60)(60)}\right)^2} = \pm 0.15s$$

$$t = 10\min = 600s$$

$$\Rightarrow u_N = \pm\sqrt{u_1^2 + u_c^2} = \pm\sqrt{0.15^2 + \left(10(600)\frac{60}{1(30)(24)(60)(60)}\right)^2} = \pm 0.151s$$

Example

- First stage uncertainty is more important (bigger) than the instrument uncertainty. For what duration is this true?

$$u_c = \pm\frac{60s}{1(30)(24)(60)(60)s}T$$

$$u_1 = \pm 0.150s$$

$$T = ???$$

$$\Rightarrow u_N = \pm\sqrt{u_1^2 + u_c^2} = \pm\sqrt{0.15^2 + \left(T\frac{60}{1(30)(24)(60)(60)}\right)^2}$$

$$T\frac{60}{1(30)(24)(60)(60)} > 0.15 \Rightarrow t > 1.8hrs$$

5.8 Multiple Measurement Uncertainty

- This is applicable if we are able to obtain multiple measurement for each variable of our experiment.
- The propagation of random uncertainty:

$$P = (P_1^2 + P_2^2 + \cdots P_K^2)^{1/2}$$

standard random uncertainty of each measurement

- The propagation of elemental systematic error is:

$$B = (B_1^2 + B_2^2 + \cdots B_K^2)^{1/2}$$

systematic uncertainty of each measurement

5.8 Multiple Measurement Uncertainty

- The total uncertainty is then:

$$u_x = \pm\left(B^2 + \left(t_{v,95}P\right)^2\right)^{1/2} \quad (95\%)$$

- v is found via the Welch-Satterthwaite formula:

$$v = \frac{\left(\sum_{k=1}^{K} P_k^2\right)^2}{\sum_{k=1}^{K}\left(P_k^4 / v_k\right)}$$

5.8 Multiple Measurement Uncertainty

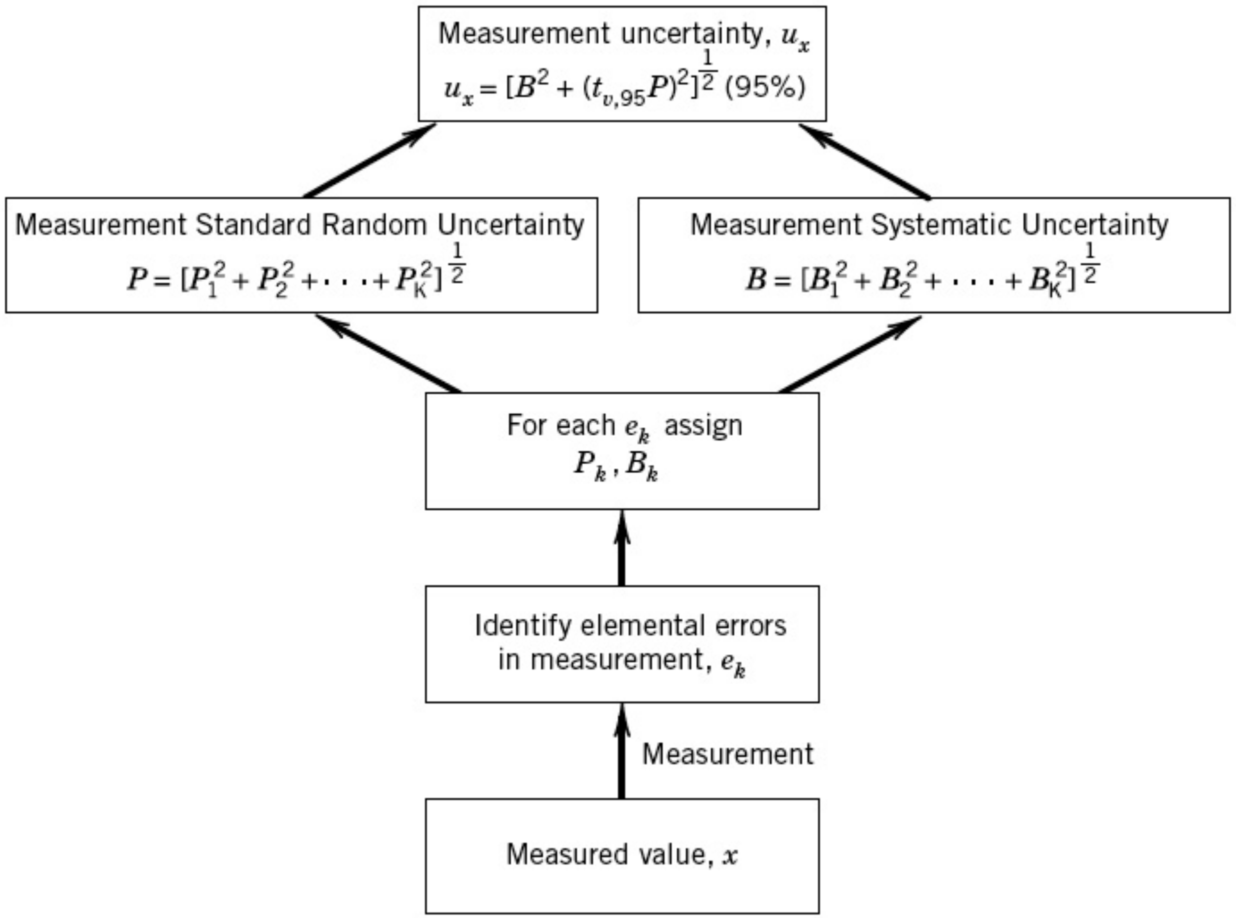


Figure 5.6. Multiple-measurement uncertainty procedure for combining uncertainties.

Example 5.12

- Measuring stress in a loaded beam we observe three sources of uncertainty as follows:

$B_1 = 1.0\text{N/cm}^2$	$B_2 = 2.1\text{N/cm}^2$	$B_3 = 0\ \text{N/cm}^2$
$P_1 = 4.6\text{N/cm}^2$	$P_2 = 10.3\text{N/cm}^2$	$P_3 = 1.2\text{N/cm}^2$
$v_1 = 14$	$v_2 = 37$	$v_3 = 8$

- If the mean value of stress is 223.4N/cm^2
- Determine the best estimate of stress at 95% confidence

Example 5.12

- The random uncertainty is:

$$P = (P_1^2 + P_2^2 + \cdots P_K^2)^{1/2} = 11.3N/cm^2$$

- The systematic uncertainty is:

$$B = (B_1^2 + B_2^2 + \cdots B_K^2)^{1/2} = 2.3N/cm^2$$

- The degree of freedom is:

$$v = \frac{\left(\sum_{k=1}^{3} P_k^2\right)^2}{\sum_{k=1}^{3}\left(P_k^4 / v_k\right)} \approx 49$$

Example 5.12

- $t_{49,95} = 2$ and the uncertainty estimate is:

$$u_\sigma = \pm[B^2 + (t_{\nu,95}P)^2]^{1/2} = \pm[2.3^2 + (2 \times 11.3)^2]^{1/2} = \pm 22.7\,N/cm^2$$

- The best estimate of a result is:

$$\sigma' = 223.4 \pm 22.7\,N/cm^2 \quad (95\%)$$

Propagation of Uncertainty

- If we have a function $f(x_1, x_2, \ldots, x_n)$, the true value can be expressed as:

$$\bar{R} = f_1\left(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_L\right)$$

$$R' = \bar{R} \pm u_R \quad (P\%)$$

$$u_R = \pm\left(B_R^2 + \left(t_{\nu,95} P_R\right)^2\right)^{1/2} \quad (95\%)$$

$$B_R = \left(\sum_{i=1}^{L}\left[\theta_i B_{x_i}\right]^2\right)^{1/2}$$

$$\nu = \frac{\left(\sum_{k=1}^{K} \theta_i P_{x_i}^2\right)^2}{\sum_{k=1}^{K}\left(\left(\theta_i P_{x_i}\right)^4 / \nu_{x_i}\right)}$$

$$P_R = \left(\sum_{i=1}^{L}\left[\theta_i P_{x_i}\right]^2\right)^{1/2}$$

Example 5.13

- Given: $\rho = \dfrac{P}{RT}$

- For the pressure measurement:

  $u_c = 1\%$ of the reading

  $N_p = 20$ measurements

  $\overline{p} = 2253.91\,psfa$

  $S_p = 167.21$ psfa

Example 5.13

- For the temperature measurement:

  $u_c = 0.6\ ^oR$

  $N_T = 10$ measurements

  $\bar{T} = 560.4\ ^oR$

  $S_T = 3.0\ ^oR$

- The gas constant $R = 54.7$ ft-lb/lbm-^oR

Example 5.13

- Find:
  - The best estimate of the density

- Assume:
  - Ideal gas
  - No uncertainty in the gas constant *R*

Example 5.13

- Solution:

$$\bar{\rho} = \frac{\bar{p}}{R\bar{T}} = 0.074\,lbm/ft^3$$

- Important:
  - Since no information is given in terms of sources of error we will only assume errors from the data-acquisition error source group; namely:
    - instrument error
    - temporal variation

Example 5.13

- The instrument error for pressure is:

$$(B_1)_p = 0.01(2253.91) = 22.5\,psfa \qquad (P_1)_p = 0$$

- The temporal variation causes:

$$(P_2)_p = S_{\bar{p}} = \frac{S_p}{\sqrt{N}} = \frac{167.21\,psfa}{\sqrt{20}} = 37.4\,psfa \qquad \nu_p = 19 \qquad (B_2)_p = 0$$

- The instrument error for temperature is:

$$(B_1)_T = 0.6\ {}^oR \qquad (P_1)_T = 0$$

- The temporal variation causes:

$$(P_2)_T = S_{\bar{T}} = \frac{S_T}{\sqrt{N}} = \frac{3.0^oR}{\sqrt{10}} = 0.9^oR \qquad \nu_p = 9 \qquad (B_2)_T = 0$$

Example 5.13

- We can now combine the find the uncertainties

$$\left(B\right)_p = \left[\left(22.5\right)^2 + \left(0\right)^2\right]^{1/2} = 22.5\, psfa$$

$$\left(P\right)_p = \left[\left(0\right)^2 + \left(37.4\right)^2 +\right]^{1/2} = 37.4\, psfa$$

- Similarly:

$$\left(B\right)_T = 0.6^o R$$

$$\left(P\right)_T = 0.9^o R$$

Example 5.13

- Now we do propagation of systematic and random uncertainties to estimate the uncertainties in density:

$$P_\rho = \left[\left(\frac{\partial\rho}{\partial T}P_T\right)^2 + \left(\frac{\partial\rho}{\partial p}P_p\right)^2\right]^{1/2} = 0.0012\,lbm/ft^3$$

- Similarly:

$$B_\rho = \left[\left(\frac{\partial\rho}{\partial T}B_T\right)^2 + \left(\frac{\partial\rho}{\partial p}B_p\right)^2\right]^{1/2} = 0.0007\,lbm/ft^3$$

Example 5.13

- Finally we have to find the equivalent # DoF

$$\nu = \frac{\left(\sum_{k=1}^{K} \theta_i P_{x_i}^2\right)^2}{\sum_{k=1}^{K}\left(\left(\theta_i P_{x_i}\right)^4 / \nu_{x_i}\right)} = \frac{\left[\left(\frac{\partial \rho}{\partial T} P_T\right)^2 + \left(\frac{\partial \rho}{\partial p} P_p\right)^2\right]^2}{\left(\frac{\partial \rho}{\partial T} P_T\right)^4 / \nu_T + \left(\frac{\partial \rho}{\partial p} P_p\right)^4 / \nu_p} = 23$$

- And the total uncertainty becomes:

$$u_\rho = \left[B_\rho^2 + \left(t_{23,95} P_\rho\right)^2\right] = 0.0025 lbm / ft^3 \quad (95\%)$$

$$\rho = 0.074 \pm 0.0025 lbm / ft^3 \quad (95\%)$$