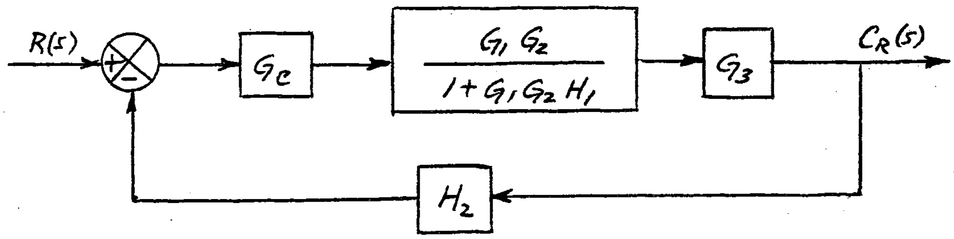
Homework # 1 Solution (Exercises 3.7, 8, 9, 11 & 14)

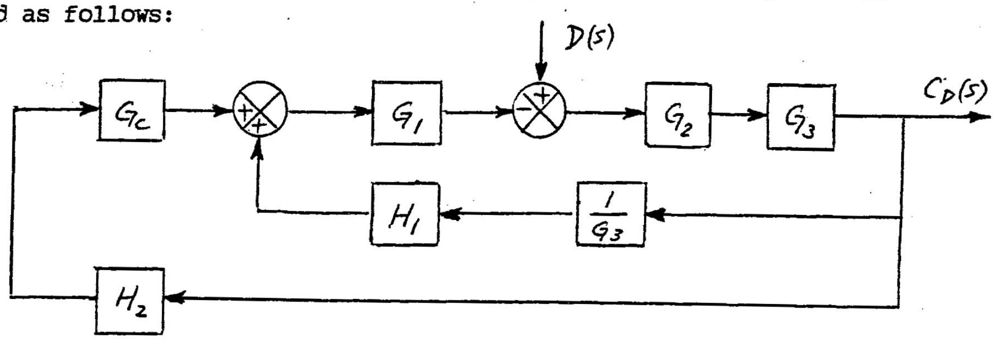
B-3-7. When D(s) = 0, the block diagram of the system can be simplified as follows:

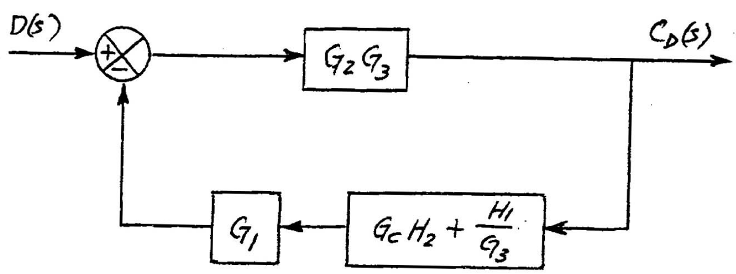


The closed-loop transfer function $C_R(s)/R(s)$ can be given by

$$\frac{C_R(s)}{R(s)} = \frac{\dfrac{G_c G_1 G_2 G_3}{1+G_1 G_2 H_1}}{1+\dfrac{G_c G_1 G_2 G_3 H_2}{1+G_1 G_2 H_1}} = \frac{G_c G_1 G_2 G_3}{1+G_1 G_2 H_1 + G_c G_1 G_2 G_3 H_2}$$

When R(s) = 0, the block diagram of the system shown in Figure 3-76 can be modified as follows:





Hence

$$\frac{C_D(s)}{D(s)} = \frac{G_2 G_3}{1+G_2 G_3 G_1 \left(G_c H_2 + \dfrac{H_1}{G_3}\right)} = \frac{G_2 G_3}{1+G_1 G_2 G_3 G_c H_2 + G_1 G_2 H_1}$$

B-3-8. There are infinitely many state-space representations for this system. We shall give two of the possible state-space representations.

State-space representation 1: From Figure 3-77, we obtain

$$\frac{Y(s)}{U(s)} = \frac{\frac{s+z}{s+p}\,\frac{1}{s^2}}{1+\frac{s+z}{s+p}\,\frac{1}{s^2}} = \frac{s+z}{s^3+ps^2+s+z}$$

which is equivalent to

$$\dddot{y} + p\ddot{y} + \dot{y} + zy = \dot{u} + zu$$

Comparing this equation with the standard equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we obtain

$$a_1 = p, \quad a_2 = 1, \quad a_3 = z, \quad b_0 = 0, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = z$$

Define

$$x_1 = y - \beta_0 u$$
$$x_2 = y - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

where

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1 \beta_0 = 0$$
$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 = 1$$
$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 = z - p$$

Then, state-space equations can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z & -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

or

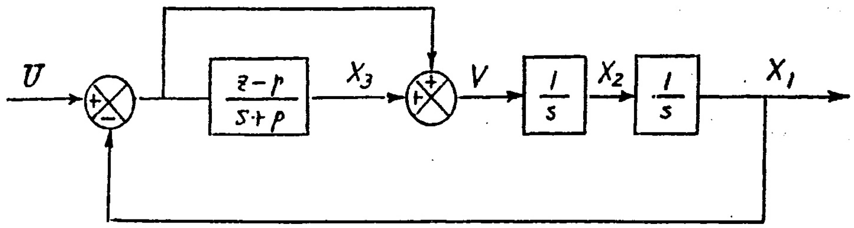
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z & -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z-p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>State-space representation 2</u>: Since

$$\frac{s+z}{s+p} = \frac{s+p+z-p}{s+p} = 1 + \frac{z-p}{s+p}$$

we can redraw the block diagram as shown below.



From this block disgram we get the following equations:

$$V = U - X_1 + X_3$$

$$\frac{X_3}{U - X_1} = \frac{z-p}{s+p}$$

$$\frac{X_2}{U - X_1 + X_3} = \frac{1}{s}$$

$$\frac{X_1}{X_2} = \frac{1}{s}$$

from which we obtain

$$\dot{x}_3 + p x_3 = (z-p)u - (z-p)x_1$$
$$\dot{x}_2 = -x_1 + x_3 + u$$
$$\dot{x}_1 = x_2$$

Rewriting, we have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_3 + u$$
$$\dot{x}_3 = -(z-p)x_1 - p x_3 + (z-p)u$$
$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ p-z & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z-p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-3-9.

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = u$$

Define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

Then

$$\dot{x}_3 + 3x_3 + 2x_2 = u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_1 = x_2$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-3-11.

$$\underset{\sim}{A} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \underset{\sim}{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The transfer function G(s) of the system is given by

$$G(s) = \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \frac{1}{(s+5)(s+1)+3} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+8} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2s-3 \\ 5s+31 \end{bmatrix} = \frac{12s+59}{s^2+6s+8}$$

A MATLAB solution to this problem is given below.

```
A=[-5 -1;3 -1];
B=[2;5];
C=[1 2];
D=0;
[num,den] = ss2tf(A,B,C,D)

num =

   0  12  59


den =

   1   6   8
```

B-3-14.

(a)

$$m\ddot{x} + kx = u$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

(b) Define the displacement of a point between springs k_1 and k_2 as y. Then the equations of motion for this system become

$$m\ddot{x} + k_2(x - y) = u$$
$$k_1 y = k_2(x - y)$$

From the second equation, we have

$$k_1 y + k_2 y = k_2 x$$

or

$$y = \frac{k_2}{k_1 + k_2} x$$

Then

$$m\ddot{x} + \frac{k_1 k_2}{k_1 + k_2} x = u$$

or

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + \dfrac{k_1 k_2}{k_1 + k_2}}$$
