# Homework # 8 Solution (Exercises 11.1, 4, 10 & 13)

**B-11-1.**

**(a) Controllable canonical form:**

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

**(b) Observable canonical form:**

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

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**B-11-4.** Referring to Equation (3-29), we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 & -1 \\ -1 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)(s+3)} \begin{bmatrix} (s+2)(s+3) & 0 & s+2 \\ s+3 & (s+1)(s+3) & 1 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+3}{(s+1)(s+2)(s+3)} = \frac{s+3}{s^3 + 6s^2 + 11s + 6}$$

Although this is a third-order system, there is a cancellation of (s + 3) in the numerator and denominator. Hence, the reduced transfer function becomes of second order.

The transfer function expression can be easily obtained from the state-space expression if MATLAB command

```
[num,den] = ss2tf(A,B,C,D)
```

is used. See the following MATLAB output.

```
A = [-1  0  1;1  -2  0;0  0  -3];
B = [0;0;1];
C = [1  1  0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

      0        0   1.0000   3.0000


den =

   1   6   11   6
```

This output corresponds to the transfer function

$$\frac{s+3}{s^3+6s^2+11s+6}$$

Notice that the MATLAB output does not show the reduced transfer function when cancellation occurs.

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**B-11-10.** A MATLAB program to obtain a state-space representation is given next.

```
num = [0  10.4  47  160];
den = [1  14  56  160];
[A,B,C,D] = tf2ss(num,den)

A =

  -14  -56  -160
    1    0     0
    0    1     0


B =

   1
   0
   0


C =

  10.4000  47.0000  160.0000


D =

   0
```

The state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10.4 & 47 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

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**B-11-13.** The controllability and observability of the system can be determined by examining the rank conditions of

$$[\underset{\sim}{B} \quad \underset{\sim}{A}\underset{\sim}{B} \quad \underset{\sim}{A}^2\underset{\sim}{B}]$$

and

$$[\underset{\sim}{C}^* \quad \underset{\sim}{A}^*\underset{\sim}{C}^* \quad (\underset{\sim}{A}^*)^2\underset{\sim}{C}^*]$$

respectively.

```
A = [-1  -2  -2;0  -1  1;1  0  -1];
B = [2;0;1];
C = [1  1  0];
D = [0];
rank([B  A*B  A^2*B])

ans =

    3

rank([C'  A'*C'  A'^2*C'])

ans =

    3
```

Since the rank of $[\underset{\sim}{B} \quad \underset{\sim}{A}\underset{\sim}{B} \quad \underset{\sim}{A}^2\underset{\sim}{B}]$ is 3 and the rank of $[\underset{\sim}{C}' \quad \underset{\sim}{A}'\underset{\sim}{C}' \quad \underset{\sim}{A}'^2\underset{\sim}{C}']$ is also 3, the system is completely state controllable and observable.

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