Homework # 9 Solution (Exercises 12.3, 8, 11 & 14)

B-12-3. Referring to Equation (12-18), the state-feedback gain matrix K can be given by

$$K = [0 \quad 0 \quad 1][B \quad AB \quad A^2B]^{-1}\phi(A)$$

where

$$\phi(A) = A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3 I$$

The values of α_1, α_2, and α_3 are determined from the desired characteristic equation:

$$|sI - (A - BK)| = (s + 2 + j4)(s + 2 - j4)(s + 10)$$
$$= s^3 + 14s^2 + 60s + 200$$
$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

Thus,

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$

Then

$$\phi(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

Since

$$\begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{A}\underset{\sim}{B} & \underset{\sim}{A}^2\underset{\sim}{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -11 \\ 1 & -11 & 60 \end{bmatrix}$$

we have the desired state-feedback gain matrix $\underset{\sim}{K}$ as follows:

$$\underset{\sim}{K} = [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -11 \\ 1 & -11 & 60 \end{bmatrix}^{-1} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [0 \quad 0 \quad 1] \begin{bmatrix} 0.7349 & 0.8554 & 0.1446 \\ 0.8554 & 0.0120 & -0.0120 \\ 0.1446 & -0.0120 & 0.0120 \end{bmatrix}$$

$$\times \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [0 \quad 0 \quad 1] \begin{bmatrix} 138.3976 & 170.2169 & 28.7831 \\ 170.2169 & 49.4819 & 5.5181 \\ 28.7831 & 5.5181 & 2.4819 \end{bmatrix}$$

$$= [28.7831 \quad 5.5181 \quad 2.4819]$$

B-12-8. From Figure 12-49 we obtain

$$u = k_1(r - x_1) - k_2 x_2 - k_3 x_3 = -\mathbf{K}\mathbf{x} + k_1 r$$

where

$$\mathbf{K} = [\,k_1 \quad k_2 \quad k_3\,]$$

Noting that the rank of

$$\mathbf{M} = [\,\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B}\,] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

is three, arbitrary pole placement is possible. The characteristic equation for this system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 5 & s+6 \end{vmatrix} = s^3 + 6s^2 + 5s$$

$$= s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Hence

$$a_1 = 6, \quad a_2 = 5, \quad a_3 = 0$$

Since the state equation for the system is already in the controllable canonical form, we have $\mathbf{T} = \mathbf{I}$. The desired characteristic equation is

$$(s + 2 + j4)(s + 2 - j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

from which we obtain

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$

Then

$$\underset{\sim}{K} = [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]\underset{\sim}{T}^{-1}$$

$$= [200-0 \quad 60-5 \quad 14-6]\underset{\sim}{I}$$

$$= [200 \quad 55 \quad 8]$$

The state equation for the designed system is

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}u = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}(-\underset{\sim}{K}\underset{\sim}{x} + k_1 r)$$

$$= (\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K})\underset{\sim}{x} + \underset{\sim}{B}k_1 r$$

Since

$$\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [200 \quad 55 \quad 8] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -60 & -14 \end{bmatrix}$$

we have

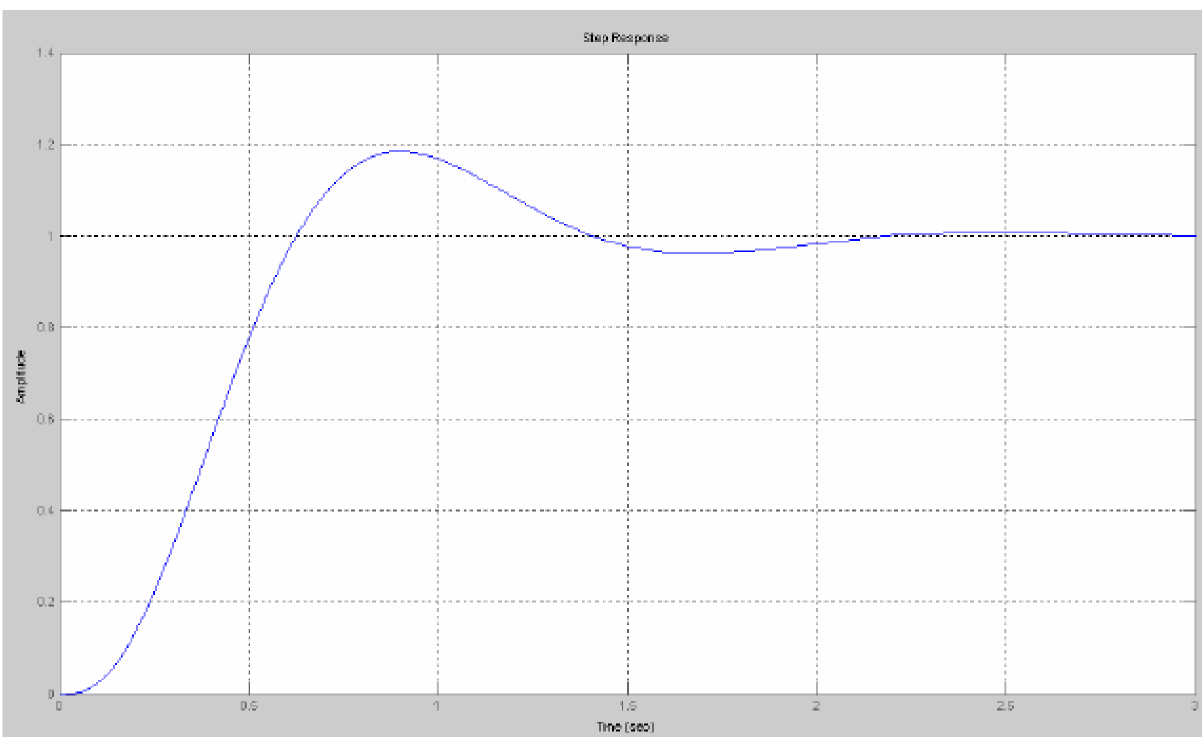
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -60 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 200 \end{bmatrix} r \tag{1}$$

The output equation is

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2}$$

The unit-step response of the designed system can be obtained from Equations (1) and (2) by substituting r = 1(t) and finding y(t). A MATLAB program to obtain the unit-step response curve [y(t) versus t curve] is given on the next page.

```
>> A = [0 1 0; 0 0 1; -200 -60 -14]
A =

     0     1     0
     0     0     1
  -200   -60   -14

>> B = [0; 0; 200]
B =

     0
     0
   200

>> C = [1 0 0]
C =

     1     0     0

>> D=[0]
D =

     0

>> step (A,B,C,D)
>> grid
```



B-12-11. A full-order state observer for the given system is designed by use of MATLAB. The MATLAB program used for the design of the state observer is given below.

```
% ***** Design of full-order state observer *****

A=[0  1  0;0  0  1;-5  -6  0];
C=[1  0  0];
L=[-10  -10  -15];
Ke=acker(A',C',L)'
Warning: Pole locations are more than 10% in error.

Ke =

    35
   394
  1285
```

Referring to Equation (12-60), the full-order state observer is given by

$$\dot{\tilde{x}} = (A - K_e C)\tilde{x} + Bu + K_e y$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} -35 & 1 & 0 \\ -394 & 0 & 1 \\ -1290 & -6 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ 394 \\ 1285 \end{bmatrix} y$$

B-12-14. We shall use the MATLAB approach to solve this problem. The first MATLAB program given in the next page determines the state feedback gain matrix $\mathbf{K}$ and the observer gain matrix $\mathbf{K}_e$. The observer to be designed is a full-order observer.

```
% ***** Determination of K and Ke *****

A = [0  1  0;0  0  1;-6  -11  -6];
B = [0;0;1];
C = [1  0  0];
J = [-1+j  -1-j  -5];
K = acker(A,B,J)

K =

    4   1   1

L = [-6  -6  -6];
Ke = acker(A',C',L)'

Ke =

   12
   25
  -72
```

The state feedback gain matrix $\mathbf{K}$ and the observer gain matrix $\mathbf{K}_e$ thus obtained are as follows:

$$\mathbf{K} = [4 \quad 1 \quad 1]$$

$$\mathbf{K}_e = \begin{bmatrix} 12 \\ 25 \\ -72 \end{bmatrix}$$

The second MATLAB program given below determines the transfer function of the observer controller.

```
% Obtaining transfer function of observer controller — full-order observer

A = [0  1  0;0  0  1;-6  -11  -6];
B = [0;0;1];
C = [1  0  0];
K = [4  1  1];
Ke = [12;25;-72];
AA = A-Ke*C-B*K;
BB = Ke;
CC = K;
DD = 0;
[num,den] = ss2tf(AA,BB,CC,DD)

num =

       0   1.0000  119.0000  618.0000


den =

    1.0000   19.0000  121.0000  257.0000
```

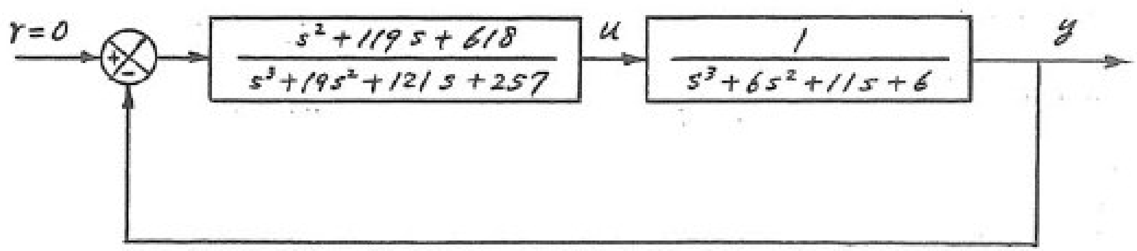
The transfer function of the observer controller is

$$\frac{U(s)}{-Y(s)} = \frac{s^2 + 119s + 618}{s^3 + 19s^2 + 121s + 257}$$

The transfer function of the given system in state space form is

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

A block diagram of the designed system is shown below.



Notice that the designed system is of sixth order.