B-12-3. Referring to Equation (12-18), the state-feedback gain matrix $\mathbf{K}$ can be given by

$$\mathbf{K} = [0 \quad 0 \quad 1] [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]^{-1} \phi(\mathbf{A})$$

where

$$\phi(\mathbf{A}) = \mathbf{A}^3 + \alpha_1 \mathbf{A}^2 + \alpha_2 \mathbf{A} + \alpha_3 \mathbf{I}$$

The values of α_1, α_2, and α_3 are determined from the desired characteristic equation:

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = (s+2+j4)(s+2-j4)(s+10)$$
$$= s^3 + 14s^2 + 60s + 200$$
$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

Thus,

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$

Then

$$\phi(\mathbf{A}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

Since

$$\begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{A}\underset{\sim}{B} & \underset{\sim}{A}^2\underset{\sim}{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -11 \\ 1 & -11 & 60 \end{bmatrix}$$

we have the desired state-feedback gain matrix $\underset{\sim}{K}$ as follows:

$$\underset{\sim}{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -11 \\ 1 & -11 & 60 \end{bmatrix}^{-1} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.7349 & 0.8554 & 0.1446 \\ 0.8554 & 0.0120 & -0.0120 \\ 0.1446 & -0.0120 & 0.0120 \end{bmatrix}$$

$$\times \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 138.3976 & 170.2169 & 28.7831 \\ 170.2169 & 49.4819 & 5.5181 \\ 28.7831 & 5.5181 & 2.4819 \end{bmatrix}$$

$$= \begin{bmatrix} 28.7831 & 5.5181 & 2.4819 \end{bmatrix}$$

B-12-5. Substituting

$$u = -\underset{\sim}{K}\underset{\sim}{x} = -[k_1 \quad k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

into the state equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \quad k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 & 1-k_2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The characteristic equation becomes

$$|s\underset{\sim}{I} - \underset{\sim}{A}| = \begin{vmatrix} s+k_1 & -1+k_2 \\ 0 & s-2 \end{vmatrix} = (s+k_1)(s-2) = 0$$

Because of the presence of one eigenvalue ($s = 2$) in the right-half s plane, the system is unstable whatever values k_1 and k_2 may assume.

B-12-8. From Figure 12-49 we obtain

$$u = k_1(r - x_1) - k_2 x_2 - k_3 x_3 = -\mathbf{K}\mathbf{x} + k_1 r$$

where

$$\mathbf{K} = [\, k_1 \quad k_2 \quad k_3 \,]$$

Noting that the rank of

$$\mathbf{M} = [\, \mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \,] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

is three, arbitrary pole placement is possible, The characteristic equation for this system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 5 & s+6 \end{vmatrix} = s^3 + 6s^2 + 5s$$

$$= s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Hence

$$a_1 = 6, \quad a_2 = 5, \quad a_3 = 0$$

Since the state equation for the system is already in the controllable canonical form, we have $\mathbf{T} = \mathbf{I}$. The desired characteristic equation is

$$(s + 2 + j4)(s + 2 - j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

from which we obtain

$$\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$$

Then

$$\underset{\sim}{K} = [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]\underset{\sim}{T}^{-1}$$
$$= [200 - 0 \quad 60 - 5 \quad 14 - 6]\underset{\sim}{I}$$
$$= [200 \quad 55 \quad 8]$$

The state equation for the designed system is

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}u = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}(-\underset{\sim}{K}\underset{\sim}{x} + k_1 r)$$
$$= (\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K})\underset{\sim}{x} + \underset{\sim}{B}k_1 r$$

Since

$$\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[200 \quad 55 \quad 8] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -60 & -14 \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -60 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 200 \end{bmatrix} r \qquad (1)$$

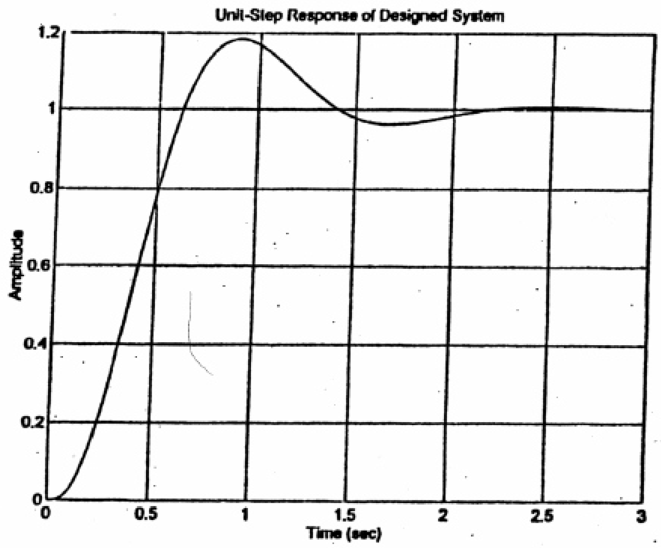
The output equation is

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (2)$$

The unit-step response of the designed system can be obtained from Equations (1) and (2) by substituting r = 1(t) and finding y(t). A MATLAB program to obtain the unit-step response curve [y(t) versus t curve] is given on the next page.

```
A = [0   1   0;0   0   1;-200   -60   -14];
B = [0;0;200];
C = [1   0   0];
D = [0];
step(A,B,C,D)
grid
title('Unit-Step Response of Designed System')
```

The resulting unit-step response curve is shown below.



B-12-12. We shall present two methods for obtaining the full-order state observer gain matrix $\underset{\sim}{K}_e$. A MATLAB solution is also given.

Method 1: Referring to Equation (12-61), the state observer gain matrix $\underset{\sim}{K}_e$ can be given by

$$\underset{\sim}{K}_e = \underset{\sim}{Q} \begin{bmatrix} \alpha_3 - a_3 \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix}$$

Matrix $\underset{\sim}{Q}$ is given by

$$\underset{\sim}{Q} = (\underset{\sim}{W}\underset{\sim}{N}^*)^{-1}$$

where

$$\underset{\sim}{N} = [\underset{\sim}{C}^* \quad \underset{\sim}{A}^*\underset{\sim}{C}^* \quad \underset{\sim}{A}^{*2}\underset{\sim}{C}^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{W} = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The values of a_1 and a_2 are determined from the characteristic equation of the original system.

$$|s\underset{\sim}{I} - \underset{\sim}{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1.244 & -0.3956 & s+3.145 \end{vmatrix}$$

$$= s^3 + 3.145s^2 - 0.3956s - 1.244$$

$$= s^3 + a_1 s^2 + a_2 s + a_3$$

Hence

$$a_1 = 3.145, \quad a_2 = -0.3956, \quad a_3 = -1.244$$

Thus

$$\underset{\sim}{W} = \begin{bmatrix} -0.3956 & 3.145 & 1 \\ 3.145 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore

$$\underset{\sim}{Q} = (\underset{\sim}{W}\underset{\sim}{N}^*)^{-1} = \begin{bmatrix} -0.3956 & 3.145 & 1 \\ 3.145 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3.145 \\ 1 & -3.145 & 10.2866 \end{bmatrix}$$

The values of α_1, α_2, and α_3 are determined from the desired characteristic equation.

$$(s-\mu_1)(s-\mu_2)(s-\mu_3)$$
$$= (s+5-j5\sqrt{3})(s+5+j5\sqrt{3})(s+10)$$
$$= (s^2+10s+100)(s+10)$$
$$= s^3+20s^2+200s+1000$$
$$= s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3 = 0$$

Thus

$$\alpha_1 = 20, \quad \alpha_2 = 200, \quad \alpha_3 = 1000$$

Hence

$$\underset{\sim}{K_e} = \underset{\sim}{Q}\begin{bmatrix} \alpha_3 - a_3 \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3.145 \\ 1 & -3.145 & 10.2866 \end{bmatrix}\begin{bmatrix} 1000+1.244 \\ 200+0.3956 \\ 20-3.145 \end{bmatrix}$$

$$= \begin{bmatrix} 16.855 \\ 147.387 \\ 544.381 \end{bmatrix}$$

Method 2: Define the state observer gain matrix as

$$\underset{\sim}{K_e} = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

The desired characteristic equation becomes

$$|s\underset{\sim}{I} - \underset{\sim}{A} + \underset{\sim}{K_e}\underset{\sim}{C}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1.244 & 0.3956 & -3.145 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix} [1 \ 0 \ 0] \right|$$

$$= \begin{vmatrix} s + k_{e1} & -1 & 0 \\ k_{e2} & s & -1 \\ -1.244 + k_{e3} & -0.3956 & s + 3.145 \end{vmatrix}$$

$$= s^3 + (k_{e1} + 3.145)s^2 + (3.145 k_{e1} + k_{e2} - 0.3956)s$$
$$+ (-1.244 + k_{e3} + 3.145 k_{e2} - 0.3956 k_{e1})$$

$$= s^3 + 20 s^2 + 200 s + 1000 = 0$$

Hence

$$k_{e1} + 3.145 = 20$$

$$3.145 k_{e1} + k_{e2} - 0.3956 = 200$$

$$-1.244 + k_{e3} + 3.145 k_{e2} - 0.3956 k_{e1} = 1000$$

from which we obtain

$$k_{e1} = 16.855 \ , \quad k_{e2} = 147.387 \ , \quad k_{e3} = 544.381$$

or

$$\underset{\sim}{K_e} = \begin{bmatrix} 16.855 \\ 147.387 \\ 544.381 \end{bmatrix}$$

Referring to Equation (12-60), the full-order state observer is

$$\dot{\tilde{\underset{\sim}{x}}} = (\underset{\sim}{A} - \underset{\sim}{K_e}\underset{\sim}{C})\tilde{\underset{\sim}{x}} + \underset{\sim}{B}u + \underset{\sim}{K_e}y$$

or

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} -16.855 & 1 & 0 \\ -147.387 & 0 & 1 \\ -543.137 & 0.3956 & -3.145 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.244 \end{bmatrix} u + \begin{bmatrix} 16.855 \\ 147.387 \\ 544.381 \end{bmatrix} y$$

MATLAB solution: A MATLAB program to obtain the state observer gain matrix $\underset{\sim}{K_e}$ is shown on the next page.

```
% ***** Design of full-order state observer *****

A = [0  1  0;0  0  1;1.244  0.3956  -3.145];
C = [1  0  0];
L = [-5+j*5*(sqrt(3))  -5-j*5*(sqrt(3))  -10];
Ke = acker(A',C',L)'

Ke =

   16.8550
  147.3866
  544.3809
```

B-12-14. We shall use the MATLAB approach to solve this problem. The first MATLAB program given in the next page determines the state feedback gain matrix $\underset{\sim}{K}$ and the observer gain matrix $\underset{\sim}{K}_e$. The observer to be designed is a full-order observer.

```
% ***** Determination of K and Ke *****

A = [0  1  0;0  0  1;-6  -11  -6];
B = [0;0;1];
C = [1  0  0];
J = [-1+j  -1-j  -5];
K = acker(A,B,J)

K =

    4   1   1

L = [-6  -6  -6];
Ke = acker(A',C',L)'

Ke =

   12
   25
  -72
```

The state feedback gain matrix $\underset{\sim}{K}$ and the observer gain matrix $\underset{\sim}{K}_e$ thus obtained are as follows:

$$\underset{\sim}{K} = [4 \quad 1 \quad 1]$$

$$\underset{\sim}{K}_e = \begin{bmatrix} 12 \\ 25 \\ -72 \end{bmatrix}$$

The second MATLAB program given below determines the transfer function of the observer controller.

```
% Obtaining transfer function of observer controller — full-order observer

A = [0  1  0;0  0  1;-6  -11  -6];
B = [0;0;1];
C = [1  0  0];
K = [4  1  1];
Ke = [12;25;-72];
AA = A-Ke*C-B*K;
BB = Ke;
CC = K;
DD = 0;
[num,den] = ss2tf(AA,BB,CC,DD)

num =

      0   1.0000  119.0000  618.0000


den =

   1.0000  19.0000  121.0000  257.0000
```

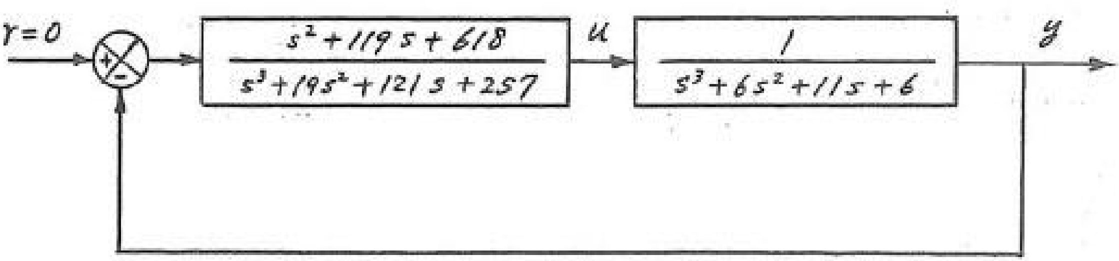
The transfer function of the observer controller is

$$\frac{U(s)}{-Y(s)} = \frac{s^2 + 119s + 618}{s^3 + 19s^2 + 121s + 257}$$

The transfer function of the given system in state space form is

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

A block diagram of the designed system is shown below.



Notice that the designed system is of sixth order.