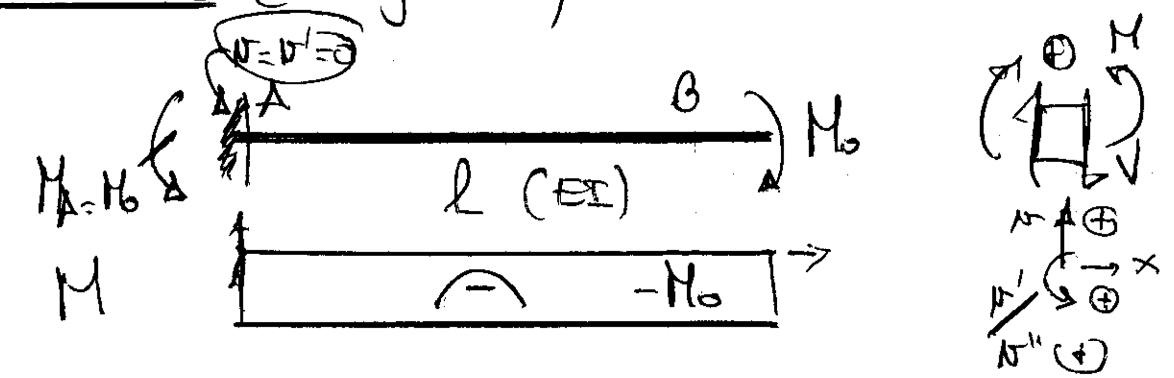
Deflection of Beams: Integration, Mom-Area, Conj. Beam
Textbook: Struct. Anal., Hibbeler, 6th Edition
Chap 8: See Question Sheet for Problems
+ Extra Problems I, II, III, IV on Question Sheet.

Problem 8.5 (Integration)

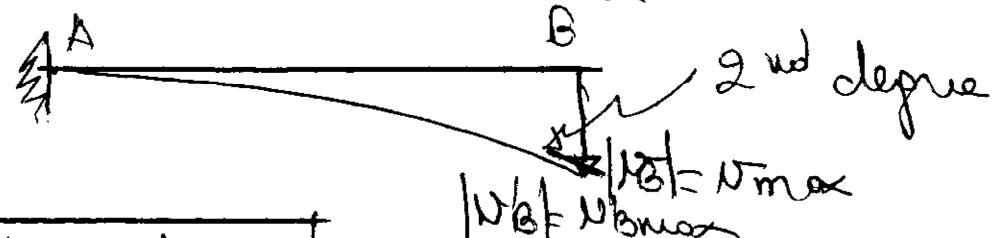


$$EI v'' = -M_0$$

$$EI v' = -M_0 X + \cancel{C_1}^{0} \quad B.C.: v'(0) = 0 \Rightarrow C_1 = 0$$

$$EI v = -\frac{M_0}{2} X^2 + \cancel{C_2}^{0} \quad B.C.\ v(0) = 0 \Rightarrow C_2 = 0$$

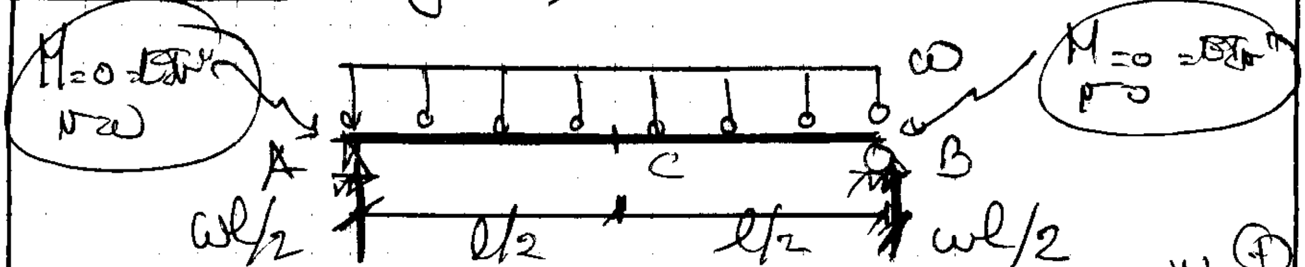
$$v = -\frac{M_0}{2EI} X^2 \quad + \quad v' \triangleq \theta = -\frac{M_0}{EI} X$$

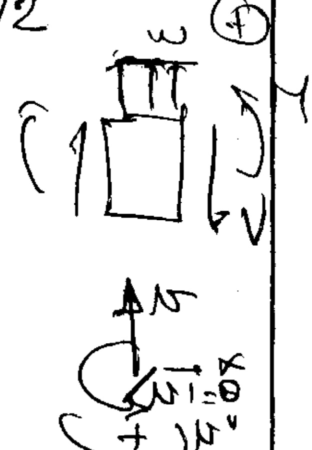


$$\boxed{(\text{Max})\ \theta_B = v'_B = (-) \frac{M_0 l}{EI} \ \text{cw}}$$

$$\boxed{(\text{Max})\ v_B = -\frac{M_0}{2EI} l^2 \ (\downarrow)}$$

Problem 8.7 (Integration)





$EI v'''' = -w$

$EI v''' = -wx + C_1$ (Shear V)

$EI v'' = -\frac{w}{2}x^2 + C_1 x + \cancel{C_2}^{\,0}$ (Mom M)

B.C.: $M(0) = EI v''(0) = 0 \Rightarrow C_2 = 0$

$M(l) = EI v''(l) = 0 \Rightarrow C_1 = \frac{wl}{2}$

$\Rightarrow EI v'' = -\frac{w}{2}x^2 + \frac{wl}{2}x$

$EI v' = -\frac{w}{6}x^3 + \frac{wl}{4}x^2 + C_3$

$EI v = -\frac{w}{24}x^4 + \frac{wl}{12}x^3 + C_3 x + \cancel{C_4}^{\,0}$

BC: $v(0) = 0 \Rightarrow C_4 = 0$

$v(l) = 0 \Rightarrow C_3 = -\frac{wl^3}{24}$

$\Rightarrow \theta = v' = -\frac{w}{6EI}x^3 + \frac{wl}{4EI}x^2 - \frac{wl^3}{24EI}$

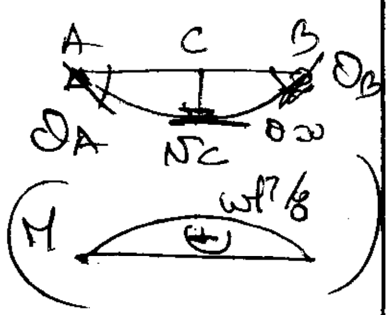
$\boxed{\max\ \theta_A = \theta(0) = \ominus \frac{wl^3}{24EI} \left(= -\theta_B\right)}$ → CW

$v = -\frac{w}{24EI}x^4 + \frac{wl}{12EI}x^3 - \frac{wl^3}{24EI}x$

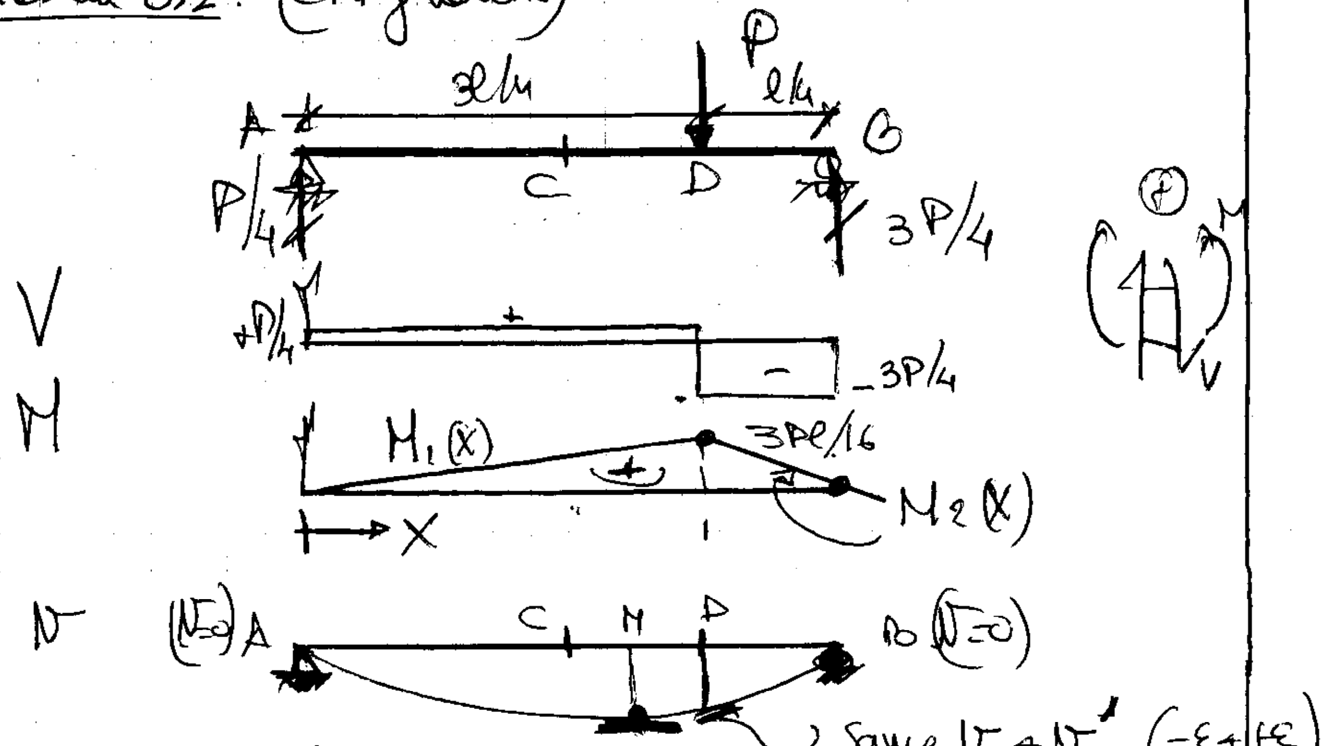
$v_{max} = v_C$ (by symmetry / inspection)

or @ $v' = 0 \Rightarrow x = l/2$

$\boxed{\max\ v_C = \ominus \frac{5}{384}\frac{wl^4}{EI}\ (\downarrow)}$



Problem 8.2: (Integration)



$$M_1(x) = \frac{Px}{4}$$

$$M_2(x) = \frac{+P}{4}x - P\left(x - \frac{3l}{4}\right)$$

(Option start x from D for M_2)

(1)	(2)
$EIv_1'' = \frac{Px}{4}$	$EIv_2'' = -\frac{3P}{4}x + \frac{3Pl}{4}$
$EIv_1' = \frac{Px^2}{8} + A_1$	$EIv_2' = -\frac{3P}{8}x^2 + \frac{3Pl}{4}x + B_1$
$EIv_1 = \frac{Px^3}{24} + A_1x + \cancel{A_2}^{\,0}$	$EIv_2 = -\frac{3P}{24}x^3 + \frac{3Pl}{8}x^2 + B_1x + B_2$

BC: $v_1(0) = 0$ (I) $\quad v_2(l) = 0$ (II)

(@D) $v_1\left(\frac{3l}{4}\right) = v_2\left(\frac{3l}{4}\right)$ (III)

$v_1'\left(\frac{3l}{4}\right) = v_2'\left(\frac{3l}{4}\right)$ (IV)

4 equations with 4 unknowns

(I) $v_1(0) = 0 \Rightarrow \boxed{A_2 = 0}$. Solve ...

$\boxed{A_1 = -\frac{5}{128}Pl^2}$ $\quad \boxed{B_1 = -\frac{41}{128}Pl^2}$ $\quad \boxed{B_2 = \frac{9}{128}Pl^3}$

$$v_1' = \theta_1 = \frac{Px^2}{8EI} - \frac{5Pl^2}{128EI} \qquad v_2' = \theta_2 = -\frac{3P}{8EI}x^2 + \frac{3Pl}{4EI}x - \frac{41Pl^2}{128EI}$$

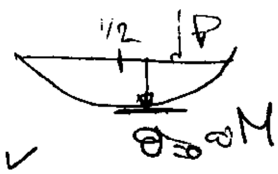
$$v_1 = \frac{Px^3}{24EI} - \frac{5Pl^2x}{128EI} \qquad v_2 = -\frac{3P}{24EI}x^3 + \frac{3Pl}{8EI}x^2 - \frac{41Pl^2}{128}x + \frac{9Pl^3}{128}$$

$$|\theta_{max}| = \theta_B = v_2'(l) = \oplus\frac{7}{128}\frac{Pl^2}{EI} \quad \oplus ccw \checkmark$$

$$[|\theta_A| < |\theta_B|]$$

$$v\left(\frac{l}{2}\right) = v_1\left(\frac{l}{2}\right) = \ominus\frac{11}{768}\frac{Pl^3}{EI} \ (\downarrow) \qquad \text{(mid point)}$$

$$v\left(\frac{3l}{4}\right) = v_1\left(\frac{3l}{4}\right) = v_2\left(\frac{3l}{4}\right) = \ominus\frac{3}{256}\frac{Pl^3}{EI} \qquad \text{(under P)}$$



v_{max} when $\theta_1 = 0$ or $v_1' = 0$

$$v_1'(0) \Rightarrow x = \pm\sqrt{0.3125\,l^2} = +0.559\,l \left(\begin{matrix}>0.5l\\<0.75l\end{matrix}\right) \checkmark$$

$$v_{max} = v_1(0.559\,l) = \ominus 0.01456\,\frac{Pl^3}{EI} \ (\downarrow)$$

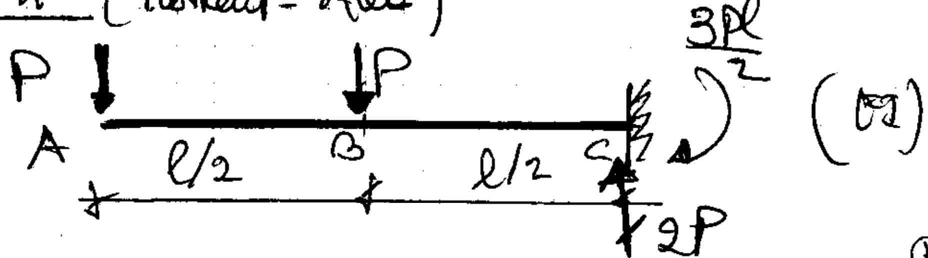
__Note:__

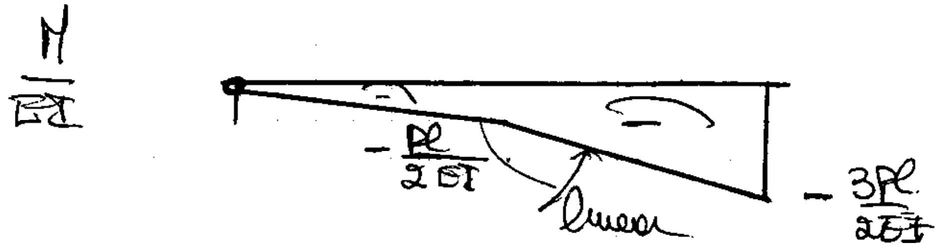
$$\left|v\left(\frac{l}{2}\right)\right| = 0.01432\,\frac{Pl^3}{EI} \ (\downarrow)$$

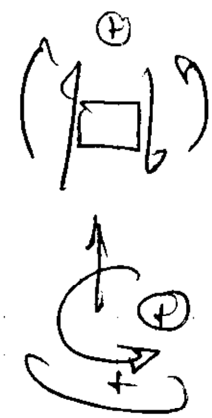
$$\left|v\left(\frac{3l}{4}\right)\right| = 0.01172 \ \text{"} \ (\downarrow)$$

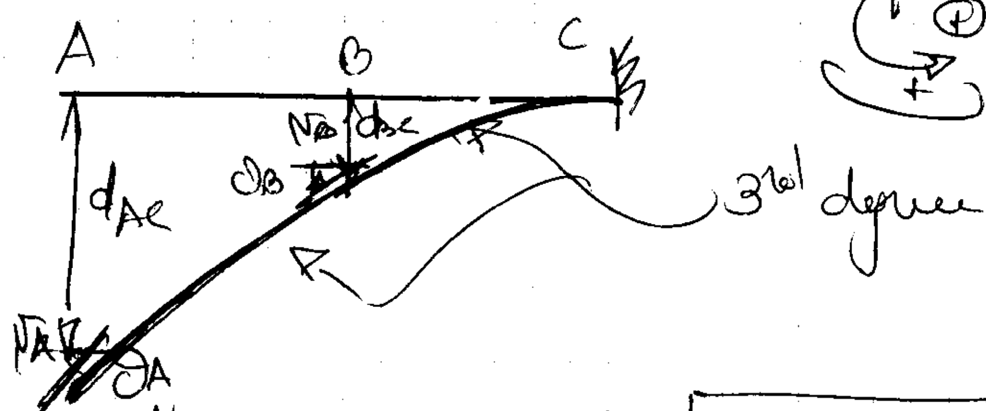
$$|v_{max}| = 0.01456 \ \text{"} \ (\downarrow) \cong v(l/2)$$

Problem 8.11 (Moment-Area)









$\theta_C^{\,0} - \theta_B = \text{area}_{l/2} = +\left(\frac{1}{2}\right)\left(-\frac{2Pl}{EI}\right)\times\left(\frac{l}{2}\right) \Rightarrow \boxed{\theta_B = +\frac{Pl^2}{2EI}\ \text{ccw}}$

$\theta_B - \theta_A = \text{area}_{l/2} \Rightarrow \theta_A = \frac{Pl^2}{2EI} - \left(\frac{1}{2}\right)\left(-\frac{Pl}{2EI}\right)\left(\frac{l}{2}\right)$

$\boxed{\theta_A = +\frac{5}{8}\frac{Pl^2}{EI}\ \text{ccw}}$

$d_{BC} = \left[-\frac{Pl}{2EI},\ l/2,\ -\frac{Pl}{EI}\right] = (1)\left(-\frac{Pl}{2EI}\right)\left(\frac{l}{2}\right)\left(\frac{l}{4}\right) + \left(\frac{1}{2}\right)\left(-\frac{Pl}{EI}\right)\left(\frac{l}{2}\right)\left(\frac{2}{3}\frac{l}{2}\right)$

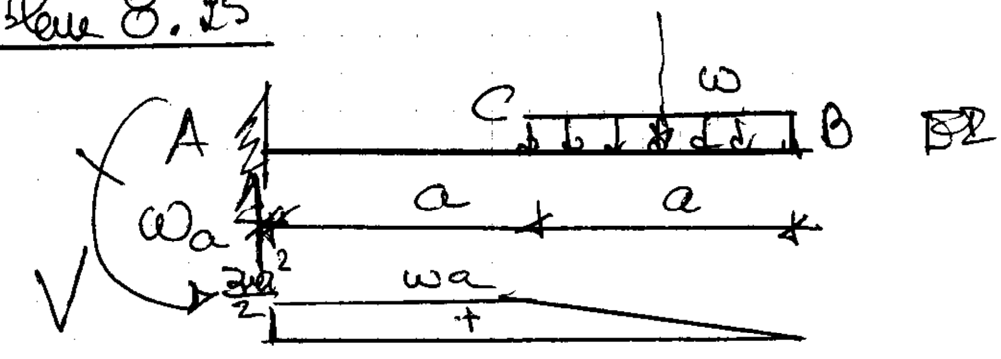
option

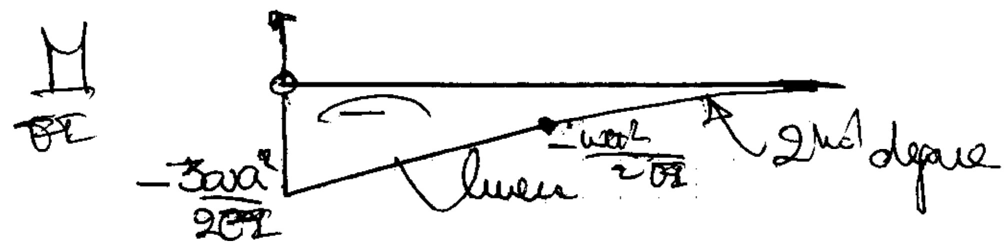
$= (-)\frac{7}{48}\frac{Pl^3}{EI}$ below $\Rightarrow \boxed{v_B = \frac{7}{48}\frac{Pl^3}{EI}\ \downarrow}$

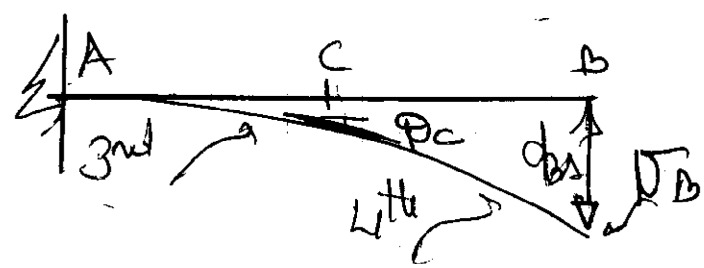
$d_{CA} = \left(l,\ -Pl/EI;\ l/2,\ l/2,\ -\frac{Pl}{2EI}\right) = \left(\frac{1}{2}\right)\left(-\frac{Pl}{EI}\right)(l)\left(\frac{2}{3}l\right) + \left(\frac{1}{2}\right)\left(-\frac{Pl}{2EI}\right)\left(\frac{l}{2}\right)\left(\frac{l}{2}+\frac{2}{3}\frac{l}{2}\right)$

$= (-)\frac{7}{16}\frac{Pl^3}{EI}$ below $\Rightarrow \boxed{v_A = \frac{7}{16}\frac{Pl^3}{EI}\ \downarrow}$

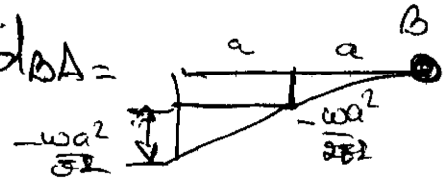
Problem 8.25







$$\theta_C - \cancel{\theta_A} = \left(-\frac{wa^2}{EI}\right)(a) = -\frac{wa^3}{EI} \quad \text{cw.} \qquad \boxed{\theta_C = -\frac{wa^3}{EI}\ \text{(cw)}}$$

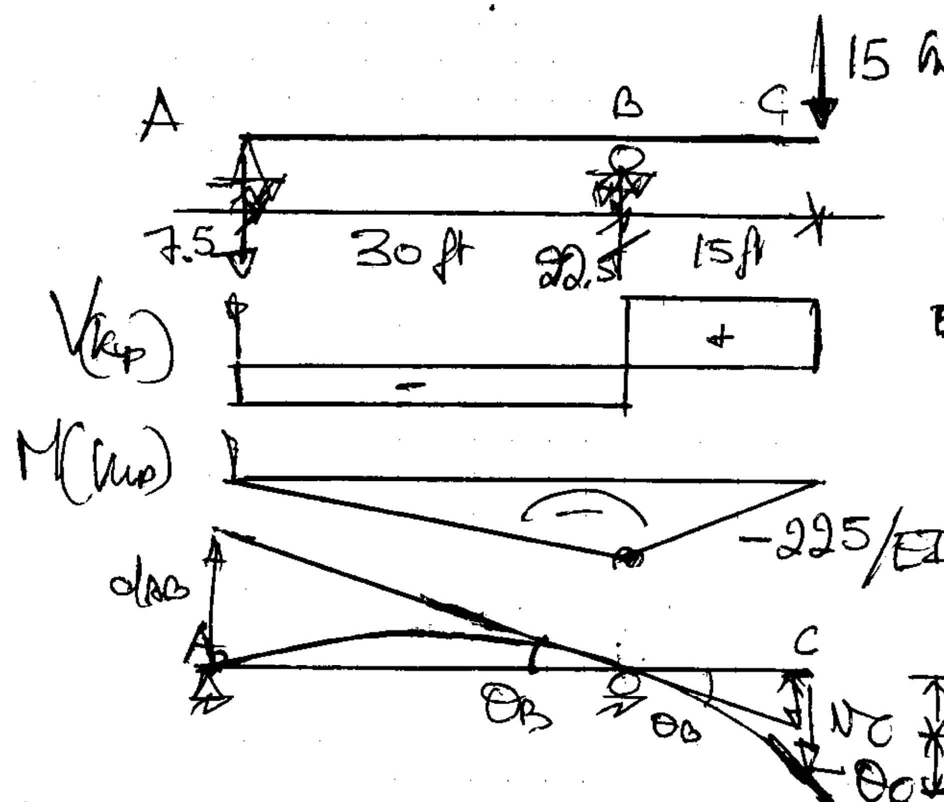


$$d_{BA} = \left(\frac{1}{3}\right)\left(-\frac{wa^2}{2EI}\right)(a)\left(\frac{3}{4}a\right) + \left(-\frac{wa^2}{2EI}\right)(a)\left(\frac{3a}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{wa^2}{EI}\right)(a)\left(a+\frac{2}{3}a\right)$$

$$= -\frac{41}{24}\frac{wa^4}{EI} \Rightarrow \boxed{v_B = \frac{41}{24}\frac{wa^4}{EI}\ \downarrow}$$

below

Problem 8.12 (Moment-Area)



$E = 29 \times 10^3$ ksi ($\times 144$) → ksf

$I = 800$ in^4 $\left(\frac{1}{12^4}\right)$ ft^4

$EI = 4,176,000$ ksf $\times \frac{800}{(12)^4} = 161,111$ k ft^2

$\Sigma M_A = 0 \Rightarrow R_B \times 30 = 15 \times 45$

$\Rightarrow R_B = 22.5$ kip (↑) $\Rightarrow R_A = 7.5$ kip (↓)

$d_{AB} = \left(\frac{1}{2}\right)\left(-\frac{225}{EI}\right) \times (30)\left(\frac{2}{3} \times 30\right) = -\frac{67,500}{EI}$

below

$|\theta_B| = \frac{67,500}{30/EI} = \frac{2250}{EI} = \frac{2,250}{161,111.1} = 0.01397$ rad CW

$|\Delta_C| = |\Delta_{CB}| + |d_{CB}|$

$d_{CB} = \left(\frac{1}{2}\right)\left(-\frac{225}{EI}\right)(15)\left(\frac{2}{3}15\right) = -16,875/EI$

below

$(\text{or } \frac{1}{2} \times d_{AB})$ ↓ $\frac{2250}{EI} \times 15 = \frac{33,750}{EI}$ $(\theta_B \times 15)$

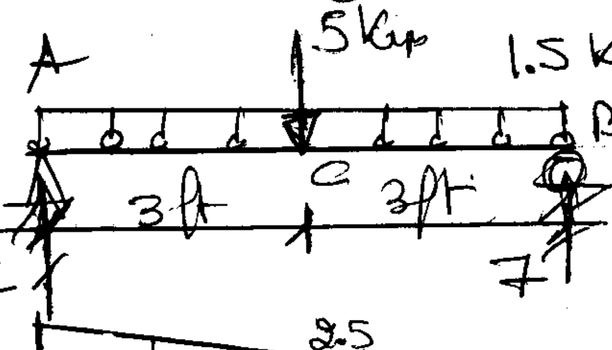
$|\Delta_C| = \frac{50,625}{EI} = 0.3142$ in

$\boxed{\Delta_C = 0.3142 \text{ ft}}$

$\theta_C - \theta_B = \Rightarrow \theta_C = \left(\frac{1}{2}\right)\left(-\frac{225}{EI}\right)(15) + \frac{2250}{EI} = -3937.5/EI$ cw

$\boxed{\theta_C = -0.0244 \text{ rad}}$

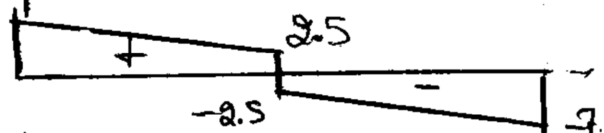
Problem 8.24 (Moment Area) + (Move by Parts)



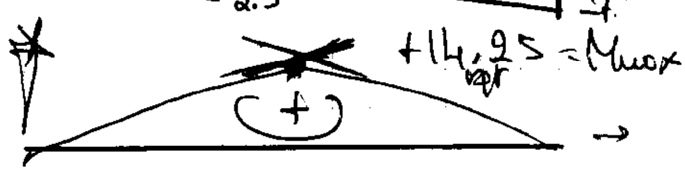
$R_A = R_B$ (↑)
$= \frac{5}{2} + \frac{1.5 \times 6}{2}$
$= 7$ kip

$E = 29{,}000$ ksi
$I = 76.8$ in^4
$EI = 15{,}466.67$ kft^2

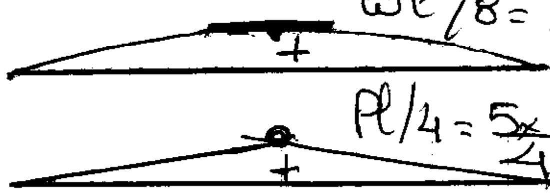
V



M

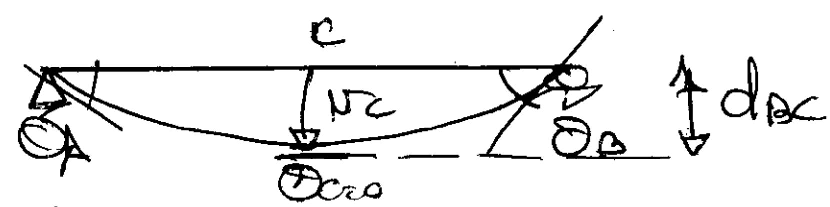


Moment by Superposition



$wl^2/8 = \frac{1.5 \times 6^2}{8} = 6.75$ kft

$Pl/4 = \frac{5 \times 6}{4} = 7.5$ kft.



Symmetry $|\theta_B| = |\theta_A|$ or $\theta_C = 0$

$$\theta_B - \cancel{\theta_C} = \underset{3}{\text{area}} + \underset{3}{\text{area}} = \left(\frac{2}{3}\right)\overbrace{\left(\frac{6.75}{EI}\right)(3)}^{13.5} + \frac{1}{2}\overbrace{\left(\frac{7.5}{EI}\right)(3)}^{11.25}$$

$$\theta_B = \frac{24.75}{EI} \approx 1.60 \times 10^{-3} \text{ rad}$$ c.cw

$\boxed{\theta_B = 1.6 \times 10^{-3} \text{ rad} \quad \text{ccw}}$

$$d_{BC} = \text{area} \cdot \tfrac{5}{8}\times 3 + \text{area} \cdot \tfrac{2}{3}\times 3 = \left(\frac{13.5}{EI}\right)\times\left(\frac{5}{8}\times 3\right) + \left(\frac{11.25}{EI}\right)\left(\frac{2}{3}\times 3\right)$$

$$\frac{25.3}{EI} + \frac{22.5}{EI} = \frac{47.8}{EI} \text{ deflection}$$

$\boxed{\Delta_C = 3.091 \times 10^{-3} \text{ ft} \downarrow}$