MECH 411 Introduction to Fluid Mechanics — Dec 25, 2004
American University of Beirut, Fall 2004-2005 — Handout # PS5

Problem Set 5 Solution

Textbook: 5.30, 5.38, 5.45, 5.67, 3.150, 3.162, 3.175, 3.176

Problem 5.30

Variables: $[\tau] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-2},\ [\rho] = \mathcal{M}\mathcal{L}^{-3},\ [\mu] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1},\ [\Omega] = \mathcal{T}^{-1},\ [R] = \mathcal{L},\ [\Delta r] = \mathcal{L}$.
Wall shear stress $\tau_w = f(\rho, \mu, \Omega, R, \Delta r)$. Using (ρ, Ω, R) as repeating variables.

Number of dimensionless groups = 6 - 3 = 3.

$$\begin{aligned}
\Pi_1 &= \frac{\tau_w}{\frac{1}{2}\rho\Omega^2 R^2} = C_f \\
\Pi_2 &= \frac{\Delta r}{R} \\
\Pi_3 &= \frac{\rho\Omega R^2}{\mu} = \mathrm{Re}
\end{aligned}$$

Therefore, in dimensionless form,

$$\frac{\tau_w}{\frac{1}{2}\rho\Omega^2 R^2} = \mathtt{f}\left(\frac{\Delta r}{R}, \frac{\rho\Omega R^2}{\mu}\right)$$

Problem 5.38

Variables: $[d] = \mathcal{L},\ [D] = \mathcal{L},\ [U] = \mathcal{L}\mathcal{T}^{-1},\ [\rho] = \mathcal{M}\mathcal{L}^{-3},\ [\mu] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1},\ [Y] = \mathcal{M}\mathcal{T}^{-2}$.

The number of dimensionless groups is $6 - 3 = 3$. Using D, ρ and U as the repeating variables, the following dimensionless groups are obtained:

$$\begin{aligned}
\Pi_1 &= \frac{\rho U D}{\mu} = \mathrm{Re} \\
\Pi_2 &= \frac{d}{D} \\
\Pi_3 &= \frac{\rho U^2 D}{Y} = \mathrm{We}
\end{aligned}$$

Therefore, the relation in dimensionless form is

$$\frac{d}{D} = \mathtt{f}\left(\mathrm{Re}, \mathrm{We}\right)$$

Problem 5.45
Model differential equation for chemical reaction dynamics in a plug reactor is

$$u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} - kC - \frac{\partial C}{\partial t}$$

(a) The above equation is physically homogeneous so that all terms have the units of s^{-1}. Then k has the unit of s^{-1} and D has the unit of m^2/s.

(b) Defining $u^* = u/V$, $x^* = x/L$, and $t^* = t/(L/V)$ then the above equation is written as

$$\frac{V}{L}u^*\frac{\partial C}{\partial x^*} = \frac{D}{L^2}\frac{\partial^2 C}{\partial x^{*2}} - kC - \frac{V}{L}\frac{\partial C}{\partial t^*}$$
$$\Rightarrow \quad u^*\frac{\partial C}{\partial x^*} = \frac{D}{VL}\frac{\partial^2 C}{\partial x^{*2}} - \frac{kL}{V}C - \frac{\partial C}{\partial t^*}$$

The pi groups that appear are $\frac{D}{VL}$ which is ratio of mass diffusion effect to inertia effect and kL/V which is the ratio of reaction time scale to inertia time scale.

Problem 5.67

Drag $\mathcal{D}$ on prototype of characteristic dimension d_p moving at velocity U_p in air at standard atmospheric conditions.
Model has characteristic dimension d_m s.t. $d_p/d_m = f$.

Under dynamically similar conditions, is $\mathcal{D}_p = \mathcal{D}_m$?

For flow conditions to be dynamically similar between model and prototype, all dimensionless variables must be equal, i.e.

$$\Pi_m = \Pi_p$$

where Π is a dimensionless group. The variables of interest in the problem are $\mathcal{D}, U, d, \rho, \mu$ where ρ and μ are density and viscosity of air. Using Pi theorem, or by inspection, we can express the drag in dimensionless form as follows

$$\frac{\mathcal{D}}{\frac{1}{2}\rho U^2 d^2} = \mathtt{f}\left(\frac{\rho U d}{\mu}\right)$$

Then we must have

$$\left\{\frac{\rho U d}{\mu}\right\}_m = \left\{\frac{\rho U d}{\mu}\right\}_p \Rightarrow \frac{U_p}{U_m} = \frac{1}{f}$$

$$\left\{\frac{\mathcal{D}}{\frac{1}{2}\rho U^2 d^2}\right\}_m = \left\{\frac{\mathcal{D}}{\frac{1}{2}\rho U^2 d^2}\right\}_p \Rightarrow \frac{\mathcal{D}_p}{\mathcal{D}_m} = 1$$

So the drag force on the model is identical to that on the prototype. Note that a necessary condition is that the flow must be similar in other aspects such as compressibility.

Problem 5.73

Power generated by a certain windmill is $P = f(D, \rho, V, \Omega, n)$ where D is diameter, ρ is air density, Ω is rotation rate, and n is the number of blades.

The number of dimensionless groups is $6 - 3 = 3$. We choose the repeating variables to be ρ, V, and D. The dimensionless groups are

$$\begin{aligned} \Pi_1 &= \frac{P}{\rho V^3 D^2} \\ \Pi_2 &= \frac{\Omega D}{V} \\ \Pi_3 &= n \end{aligned}$$

Then in dimensionless form,

$$\frac{P}{\rho V^3 D^2} = \mathtt{f}\left(\frac{\Omega D}{V}, n\right)$$

For geometrical and dynamical similarity between model and prototype we must have

$$\left\{\frac{P}{\rho V^3 D^2}\right\}_m = \left\{\frac{P}{\rho V^3 D^2}\right\}_p \Rightarrow P_p = P_m \left(\frac{V_p}{V_m}\right)^3 \left(\frac{D_p}{D_m}\right)^2$$

Problem 3.150

In a reference frame moving with the airfoil, and on a streamline form a point far ahead of the airfoil to a point above the wing, Bernoulli's equation gives

$$p_a + \frac{1}{2}\rho U_0^2 = p_t + \frac{1}{2}\rho U_t^2$$

$$\Rightarrow \quad p_t = p_a + \frac{1}{2}\rho\left(U_0^2 - U_t^2\right)$$

Then the average pressure on top of the wing is

$$\bar{p}_t = p_a + \frac{1}{2}\rho\left(U_0^2 - \bar{U}_t^2\right)$$

Similarly the average pressure on the bottom of the airfoil is

$$\bar{p}_b = p_a + \frac{1}{2}\rho\left(U_0^2 - \bar{U}_b^2\right)$$

The lift is given by

$$\mathcal{L} = (p_b - p_t)\; w\, l\, \cos\alpha = \frac{1}{2}\rho\left(\bar{U}_t^2 - \bar{U}_b^2\right)\; w\, l\, \cos\alpha$$

For a standard atmosphere the density of air varies with elevation according to

$$\rho = \rho_0\left(1 - \frac{z}{44329}\right)^{4.2558}$$

where $\rho_0 = 1.225$ kg/m^3. For $z = 5000$ m, $\rho = 0.736$ kg/m^3. For $\bar{U}_t = 215$ m/s and $\bar{U}_b = 185$ m/s, $wl = 1.5 \times 18$ m^2, then $\mathcal{L} = 119$ kN.

Problem 3.162

If the volumetric flow rate and the required pressure rise per hole are respectively Q and Δp, then total required volume flow rate and pressure drop required by the blower are respectively $Q_{tot} = n\,Q$ and $\Delta p_{tot} = n\,\Delta p$ where n is the number of holes in the table.

The volume flow rate per hole is $Q = V\frac{\pi D^2}{4}$ where D is the hole diameter. For $n = 2592$, $D = 1/16$ in, $V = 50$ ft/s, we get $Q_{tot} = 2.76$ ft^3/s.

Assuming that the air is stagnant in the large manifold under the table surface with pressure p_0, then applying Bernoulli's equation along a streamline from the stagnant region below the table surface to the exit of a hole at which the pressure is atmospheric and the velocity is $V = 50$ ft/s, and neglecting gravity effects,

$$p_0 = p_a + \frac{1}{2}\rho V^2$$

The pressure rise (from atmospheric) required by the blower $\Delta p = \rho V^2/2$. For $\rho = 0.07648$ lb/ft^3, we get $\Delta p = 0.0206$ lbf/in^2.

Problem 3.175

First we apply integral form of conservation of mass for the control volume shown in Figure:

$$V_1 H = V_2(H - h - \Delta h) \tag{1}$$

Next, we apply Bernoulli's equation along a streamline on the surface from (1) to (2) as shown in Figure,

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

Noting that $p_1 = p_2 = p_a$, then

$$V_1^2 = V_2^2 - 2g\Delta h \tag{2}$$

Combining Eqs (1) and (2) leads to

$$\begin{aligned} V_1^2 &= V_1^2 \frac{H^2}{(H-h-\Delta h)^2} - 2g\Delta h \\ \Rightarrow \quad V_1^2 &= 2g\Delta h \left[\frac{H^2}{(H-h-\Delta h)^2} - 1\right]^{-1} \\ \Rightarrow \quad Q_1 = V_1 H &= \sqrt{2g\Delta h}\, H \left[\frac{H^2}{(H-h-\Delta h)^2} - 1\right]^{-2} \end{aligned}$$

For $g = 9.8$ m/s^2, $H = 2$ m, $\Delta h = 0.1$ cm, $h = 0.3$ m, we get $Q = 8.85$ m^3/s/m.

Problem 3.176

Applying conservation of mass for the control volume shown in Figure,

$$V_1 H_1 = V_2 H_2 \tag{3}$$

Applying Bernoulli's equation between (1) and (2) on a streamline on the surface as shown in Figure, and noting that $p_1 = p_2 = p_a$, then

$$\begin{aligned} & p_a + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_a + \frac{1}{2}\rho V_w^2 + \rho g z_2 \\ \Rightarrow \quad & V_2^2 = V_1^2 + 2g(H_1 - H_2) \end{aligned} \tag{4}$$

Combining Eqs (3) and (4) leads to

$$V_2 = \left[2g \frac{H_1^2}{H_1 + H_2} \right]^{1/2}$$

For $H_1 = 5$ m, $H_2 = 0.7$ m, we get $V_2 = 9.27$ m/s.

To determine the force on the spillway we apply conservation of momentum using the control volume shown in Figure. For steady incompressible flow with uniform flow and inlet and outlet, the x component of the integral form of the conservation of momentum (per unit width) is

$$-F_{spill} + \int_0^{H_1} p_1 \, dz - \int_0^{H_2} p_2 \, dz = \dot{m} \left(V_2 - V_1 \right)$$

If the flow is hydrostatic at (1) and (2), then the gage pressures $p_1 = \rho g (H_1 - z)$ and $p_2 = \rho g (H_2 - z)$, so that

$$\begin{aligned} F_{spill} &= \int_0^{H_1} \rho g (H_1 - z) \, dz - \int_0^{H_2} \rho g (H_2 - z) \, dz + \dot{m} \left(V_1 - V_2 \right) \\ &= \frac{1}{2} \rho g \left(H_1^2 - H_2^2 \right) + \rho V_2^2 H_2 \left(\frac{H_2}{H_1} - 1 \right) \\ &= \frac{1}{2} \rho g \left(H_1^2 - H_2^2 \right) - 2 \rho g H_1 H_2 \frac{H_1 - H_2}{H_1 + H_2} \end{aligned}$$

For $\rho = 1000$ kg/m^3, $g = 9.8$ m/s^2, $H_1 = 5$ m, $H_2 = 0.7$ m, $V_2 = 9.27$ m/s, we get $F_{spill} = 63.8$ kN/m.