MECH 411 Introduction to Fluid Mechanics
American University of Beirut, Fall 2004-2005

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Handout # PS4

Problem Set 4 Solution

Textbook: 4.17, 4.21, 4.37, 4.41, 4.43, 4.53, 4.60, 4.83

Problem 4.17

(a) The flow is two-dimensional, incompressible, and steady. Given $u(x, y)$ we are asked to find $v(x, y)$. We employ the continuity (conservation of mass) equation in differential form

$$\frac{D\rho}{Dt} + \mathbf{u} \cdot \nabla \mathbf{u} = 0$$

For incompressible 2D flow, we get

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
\Rightarrow \quad &\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \delta}\frac{\partial \delta}{\partial x} \\
\Rightarrow \quad &\frac{\partial v}{\partial y} = -U\left(-\frac{2y}{\delta^2} + \frac{2y^2}{\delta^3}\right)\frac{1}{2}Cx^{-1/2}
\end{aligned}$$

Integrating from $y = 0$ to y and noting that $\delta = \delta(x)$ and using the no slip condition: $v(y = 0) = 0$, then

$$\begin{aligned}
&\int_0^y \frac{\partial v}{\partial y}\, dy = \int_0^y -U\left(-\frac{2y}{\delta^2} + \frac{2y^2}{\delta^3}\right)\frac{1}{2}Cx^{-1/2}\, dy \\
\Rightarrow \quad &v(x, y) = -U\left(-\frac{y^2}{\delta^2} + \frac{2y^3}{3\delta^3}\right)\frac{1}{2}Cx^{-1/2} \\
\Rightarrow \quad &v(x, y) = U\left(\frac{y^2}{\delta^2} - \frac{2y^3}{3\delta^3}\right)\frac{1}{2}\frac{\delta}{x}
\end{aligned}$$

It can be shown that v is maximum where $\partial u/\partial y = 0$ leading to $y^* = \delta$ so that

$$v_{max} = \frac{U}{6}\frac{\delta}{x}$$

For $U = 3$ m/s, $\delta = 1.1$ cm, and $x = 1$ m, we get $v_{max} = 0.0055$ m/s.

Problem 4.21

The flow is steady and 1D. The speed of sound is $a_s = 340$ m/s. The objective is to find $(D_e/D_0)_{min}$ to neglect compressibility effects for (a) $V_0 = 10$ m/s and (b) $V_0 = 30$ m/s.

The incompressibility assumption is valid for $\mathcal{M} < 0.3$, where the Mach number is $\mathcal{M} \equiv V/a_s$. We assume that the flow is incompressible and later we find the condition that has to be satisfied. So for an incompressible, steady, 1D flow the continuity equation reduces to

$$\rho_0 V_0 A_0 = \rho_e V_e A_e$$

with $\rho_0 = \rho_e$, $A_0 = \pi D_0^2/4$, $A_e = \pi D_e^2/4$, then

$$V_0 D_0^2 = V_e D_e^2$$

Assume that the speed of sound does not change (what does that mean for an ideal gas knowing that $a_s = \sqrt{\gamma R T}$?) then

$$\left(\frac{D_e}{D_0}\right)^2 = \frac{V_0}{a_s \mathcal{M}_e}$$

Incompressibility implies that $\mathcal{M}_e < 0.3$ so that

$$\left(\frac{D_e}{D_0}\right)^2 > \frac{V_0}{0.3 a_s}$$

Therefore the minimum value of D_e/D_0 is

$$\left(\frac{D_e}{D_0}\right)_{min} = \left(\frac{V_0}{0.3 a_s}\right)^{1/2}$$

(a) With $a_s = 340$ m/s, $V_0 = 10$ m/s, we get $(D_e/D_0)_{min} = 0.313$

(b) With $a_s = 340$ m/s, $V_0 = 30$ m/s, we get $(D_e/D_0)_{min} = 0.542$

So for higher inlet speed, the outer diameter must be larger.

Problem 4.37

The flow is fully developed, i.e. $\partial w/\partial z = 0$, also $u = 0$, $v = 0$. The objective is to find $w(x)$ given that no pressure gradient is applied and the flow is driven by gravity. We assume also that the flow is incompressible (liquid) and steady.

We employ the momentum equation

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

Since u and v are zero, then we need only the z component of the momentum equation

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \rho g$$

Noting that $Dw/Dt = \partial w/\partial t + u\,\partial w/\partial x + v\,\partial w/\partial y + w\,\partial w/\partial y$. With $\partial()/\partial t = 0$, $u = 0$, $v = 0$ and $\partial w/\partial z = 0$ (fully developed), then $Dw/Dt = 0$. Also with $w = w(x)$ we get $\partial^2 w/\partial y^2 = \partial^2 w/\partial z^2 = 0$. Finally with no pressure gradient, the equation governing w reduced to

$$\mu \frac{\partial^2 w}{\partial x^2} - \rho g = 0$$

Integrating we get

$$w(x) = \frac{\rho g}{2\mu} x^2 + C_1 x + C_2$$

where the constants C_1 and C_2 are determined by the no-slip boundary conditions at the walls:
(a) $w = 0$ at $x = -h$ and
(b) $w = 0$ at $x = h$
We get $C_1 = 0$ and $C_2 = -\rho g/2\mu$, then

$$w(x) = \frac{\rho g}{2\mu} \left(x^2 - h^2 \right)$$

Problem 4.41

The flow is steady, incompressible with $v = w = 0$ and

$$u = 4u_{max}\frac{y(h-y)}{h^2}$$

The walls are kept at constant temperature T_w (isothermal boundary conditions.)

To get the temperature distribution, we employ the differential form of the energy equation for an ideal gas in the absence of heat sources (such as those coming from chemical reactions) and radiative heat transfer

$$\rho c_p \frac{DT}{Dt} = k\,\nabla^2 T + \Phi$$

where the thermal conductivity is (spatially) uniform and Φ is the dissipation function.

With $\partial T/\partial t = 0$ (steady), $\partial T/\partial x = 0$ (fully developed), $v = 0$, $w = 0$ we get $DT/Dt = 0$. With $T = T(y)$, $\nabla^2 T = \partial^2 T/\partial y^2$. The dissipation function (equation 4.50 in book), under the stated conditions, reduced to $\Phi = \mu(\partial u/\partial y)^2$. Then

$$\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{\partial u}{\partial y}\right)^2$$

For the given u distribution, we get

$$\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{4u_{max}}{h^2}\right)^2 (h-2y)^2$$

Integrating from 0 to y, we get

$$\frac{\partial T}{\partial y} = -\frac{\mu}{k}\left(\frac{4u_{max}}{h^2}\right)^2 \left(h^2 y - h y^2 + \frac{4}{3}y^3\right) + C_1$$

Integrating again, we get

$$T(y) = -\frac{\mu}{k}\left(\frac{4u_{max}}{h^2}\right)^2 \left(\frac{h^2}{2}y^2 - \frac{h}{3}y^3 + \frac{1}{3}y^4\right) + C_1 y + C_2$$

The constant C_1 and C_2 are determined from the isothermal boundary conditions,
(a) $T = T_w$ at $y = 0$ and
(b) $T = T_w$ at $y = h$
leading to

$$\begin{aligned} C_2 &= T_w \\ C_1 &= \frac{\mu}{k}\left(\frac{4u_{max}}{h^2}\right)^2 \frac{h^3}{2} \end{aligned}$$

Then

$$T(y) = T_w - \frac{\mu}{k}\left(\frac{4u_{max}}{h^2}\right)^2 \left(\frac{h^2}{2}y^2 - \frac{h}{3}y^3 + \frac{1}{3}y^4 - \frac{h^3}{2}y\right)$$

Problem 4.43

The boundary conditions are as follows :
(a) no-slip at the wall, i.e. $u = 0$ at $y = 0$ and
(b) free surface boundary condition at $y = h$, i.e. $u_{liquid}(y = h) = u_{air}(y = h)$ and $\tau_{liquid}(y = h) = \tau_{air}(y = h)$. With $\tau(y = h) = \mu \, \partial u/\partial y|_{y=h}$, then the shear boundary condition becomes

$$\mu_{liquid} \left. \frac{\partial u_{liquid}}{\partial y} \right|_{y=h} = \mu_{air} \left. \frac{\partial u_{air}}{\partial y} \right|_{y=h}$$

Because $\mu_{air}/\mu_{liquid} << 1$, then free surface boundary condition reduces to

$$\left. \frac{\partial u}{\partial y} \right|_{y=h} = 0$$

Problem 4.53

The problem is to determine the stream function for a fully developed steady Poiseuille pipe (axisymmetric) flow with the famous velocity distribution (in the streamwise z direction)

$$v_z = -\frac{1}{4\mu}\frac{dp}{dx}\left(R^2 - r^2\right)$$

To determine the stream function we proceed with the continuity equation in axisymmetric coordinated (r, z)

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right) + \frac{\partial}{\partial z}v_z = 0$$

rearranging

$$\frac{\partial}{\partial r}\left(rv_r\right) + \frac{\partial}{\partial z}\left(rv_z\right) = 0$$

The stream function is defined such that the continuity equation is satisfied. So choosing

$$\begin{aligned} rv_r &= -\frac{\partial \psi}{\partial z} \\ rv_z &= \frac{\partial \psi}{\partial r} \end{aligned}$$

Substituting $v_r = 0$ and v_z as given above, we get

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= 0 \\ \frac{\partial \psi}{\partial r} &= -\frac{1}{4\mu}\frac{dp}{dx}\left(R^2 r - r^3\right) \end{aligned}$$

Integrating we get

$$\psi = -\frac{1}{4\mu}\frac{dp}{dx}\left(R^2\frac{r^2}{2} - \frac{r^4}{4}\right)$$

The volume flowrate is given by

$$Q = \int_0^R d\psi = \psi|_{r=R} - \psi|_{r=0} = -\frac{1}{4\mu}\frac{dp}{dx}\frac{R^4}{4}$$

Noting that $v_{z,max} = -\frac{1}{4\mu}\frac{dp}{dx}R^2$ and the flow area is $A = \pi R^2$ then

$$\frac{Q}{A} = \frac{1}{4\pi}v_{z,max}$$

Problem 4.60

The velocity field describing the flow is $v_r = 0$, $v_z = 0$ and $v_\theta = \frac{kR^2}{r}$. The question is to find out if the flow is irrotational and to find the height at $r = R$.

To find out if the flow is irrotational we find the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. If $\boldsymbol{\omega} = \mathbf{0}$ then the flow is irrotational. In cylindrical coordinates (r, θ, z),

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r} \frac{v_r}{\partial \theta} \right) \hat{\mathbf{z}} + \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{\boldsymbol{\theta}}$$

Substituting $v_r = 0$, $v_\theta = kR^2/r$ and $v_z = 0$ results in $\boldsymbol{\omega} = \mathbf{0}$. The flow is irrotational. Even though it is not clearly stated in the problem, the flow is also inviscid (otherwise there will be vorticity generated at the wall due to viscous effects.) Since the flow is irrotational and inviscid, we can apply Bernoulli's equation between any two points. We conveniently choose the two points to be on the surface: point 1 at $(r, z = h(r))$ and the other at $(r = R, z = z_C)$, then

$$p_1 + \rho \frac{V_1^2}{2} + \rho g z_1 = p_2 + \rho \frac{V_2^2}{2} + \rho g z_2$$

where $p_1 = p_2 = p_a$, $V_1 = KR^2/r$, $z_1 = h(r)$, $V_2 = KR$, $z_2 = z_C$, then

$$h(r) - z_C = \frac{K^2 R^2}{2g} \left(1 - \frac{R^2}{r^2} \right)$$

If we selected $r >> R$ at which $h = H$, then

$$H - z_C = \frac{K^2 R^2}{2g}$$

Problem 4.83

This is a taste of classical lubrication theory. The gap h is very small compared to characteristic dimension in the flow direction, $h << L$. Furthermore, the flow is inertia free $D\mathbf{u}/Dt \simeq \mathbf{0}$. This is because the flow is (i) steady, (ii) 2D with $w = 0$, (iii) from order of magnitude analysis of the continuity equation $v \sim u(h/L) \Rightarrow v << u$ (since $h << L$). Careful analysis of the x and y components[1] of the momentum equation yields

$$\begin{aligned} 0 &\simeq -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \\ 0 &\simeq -\frac{\partial p}{\partial y} \end{aligned}$$

The second equation results in $p = p(x)$ so that the first equation becomes

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dx}$$

Integrating

$$u(y) = \frac{1}{2\mu} \frac{dp}{dx} y^2 + C_1 y + C_2$$

where the constants C_1 and C_2 are determined using the boundary conditions
(a) $u = U$ at $y = 0$ and
(b) $u = 0$ at $y = h$,
resulting in

$$\begin{aligned} C_2 &= U \\ C_1 &= -\frac{1}{2\mu} \frac{dp}{dx} h - \frac{U}{h} \end{aligned}$$

The velocity profile is then given by

$$u(y) = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + U \left(1 - \frac{y}{h}\right)$$

[1]Since the flow is 2D in x and y, the z component of the momentum equation is irrelevant.