MECH 411 Introduction to Fluid Mechanics — November 18, 2004
American University of Beirut, Fall 2004-2005 — Handout # PS3

Problem Set 3 Solution

Textbook:

Conservation of mass:
Problems 3.19, 3.23, 3.25, 3.29

Conservation of momentum:
Problems 3.39, 3.53, 3.80, 3.82

Conservation of energy:
Problems 3.132, 3.137, 3.143, 3.144

Problem 3.132

Step 1: The control volume is as shown in Figure.

Step 2: The fluid is incompressible, isothermal and steady. Assume uniform velocity and properties at inlet and exit. Additionally assume that the change in kinetic energy between 1 and 2 is small compared to change in potential energy.

Step 3: Conservation of mass leads to $u_{in} = u_{out}$ and $\dot{m}_{in} = \dot{m}_{out} = \rho\, Q$.

Apply the conservation of energy:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right) d\mathcal{V} + \int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} =$$
$$\dot{Q}_{\text{gained}} - \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

Then for the given conditions

$$\int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} = -\dot{W}_{\text{viscous}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$
$$\Rightarrow \quad \rho\, Q\, g\, (z_{out} - z_{in}) = -\rho\, Q\, g\, h_f - \dot{W}_{turbine}$$
$$\Rightarrow \quad \dot{W}_{turbine} = -\rho\, Q\, g\, \left(CQ^2\right) + \rho\, Q\, g\, H$$

The turbine work is maximum when $d(\dot{W}_{turbine})/dQ = 0$ leading to an optimal $Q_{optimal} = \sqrt{H/(3C)}$.

Problem 3.137

Step 1: The control volume is as shown in Figure.

Step 2: The fluid is incompressible, isothermal and steady. Assume uniform velocity and properties at inlet and exit.

Step 3: Conservation of mass leads to $V_{in} = V_{out} A_{out}/A_{in}$ if $A_{in}/A_{out} >> 1$, then $V_{in} \simeq 0$. Note that "in" refers to the surface and not that inlet of the suction pipe at which the pressure is unknown.

Apply the conservation of energy:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right) d\mathcal{V} + \int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} =$$
$$\dot{Q}_{\text{gained}} - \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

Then for the given conditions

$$\int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} = -\dot{W}_{\text{viscous}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$
$$\Rightarrow \quad \frac{\dot{m}}{2}\left(V_{out}^2 - V_{in}^2\right) + \dot{m}\, g\, (z_{out} - z_{in}) = -\dot{m}\, g\, h_f + \dot{W}_{pump}$$
$$\Rightarrow \quad \dot{W}_{pump} = \dot{m}\left[\frac{V_{out}^2}{2} + g\,(z_{out} + h_f - z_{in})\right]$$

With $\dot{m} = \rho\, V_{out}\, A_{out} = 167.7$ lbm/s, $V_{out} = 120$ ft/s, $g = 32.2$ ft/s^2, $\rho = 64$ lb/ft^3. Then $\dot{W}_{pump} = 40274$ lbf.ft/s so that that motor driving the pump must be $\dot{W}_{pump}/\eta_{pump} = 97$ hp.

Problem 3.143

Step 1: The control volume is as shown in Figure.

Step 2: The fluid is an ideal gas. The flow is compressible and unsteady. Assume spatially (but not temporally) uniform properties in the control volume. Neglect potential energy changes. Assume the velocity of the fluid inside the tank to be zero. Assume uniform properties at inlet and that the inlet kinetic energy is very small compared to inlet enthalpy, i.e. $V_{in}^2/2 << \hat{h}_{in}$.

Step 3: Conservation of mass leads to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} = \dot{m}_{in} \\
\Rightarrow \quad &\frac{d}{dt}(\rho \mathcal{V}) = \dot{m}_{in} \\
\Rightarrow \quad &\mathcal{V}\,\frac{d\rho}{dt} = \dot{m}_{in}
\end{aligned}$$

Apply the conservation of energy:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right) d\mathcal{V} + \int_{\mathcal{S}} \rho\left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)(\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} =$$
$$\dot{Q}_{\text{gained}} - \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

Under the stated assumptions, the energy equation reduces to

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{V}} \rho\,\hat{u}\, d\mathcal{V} - \dot{m}_{in}\,\hat{h}_{in} = 0 \\
\Rightarrow \quad &\frac{d}{dt}\int_{\mathcal{V}} \rho\, C_v T\, d\mathcal{V} - \dot{m}_{in}\,\hat{h}_{in} = 0 \\
\Rightarrow \quad &C_v\,\mathcal{V}\frac{d}{dt}(\rho\, T) - \dot{m}_{in}\,\hat{h}_{in} = 0 \\
\Rightarrow \quad &C_v\,\rho\,\mathcal{V}\frac{dT}{dt} + C_v\, T\,\mathcal{V}\frac{d\rho}{dt} - \dot{m}_{in}\, C_p\, T_{in} = 0
\end{aligned}$$

Noting that from conservation of mass $\mathcal{V}\,\frac{d\rho}{dt} = \dot{m}_{in}$, then

$$\Rightarrow \quad C_v\,\rho\,\mathcal{V}\frac{dT}{dt} = \dot{m}_{in}\, C_p\, T_{in} - \dot{m}_{in}\, C_v\, T$$

The initial rate of increase of temperature in the tank is

$$\left.\frac{dT}{dt}\right|_{t=0^+} = \frac{\dot{m}_{in}\,(C_p - C_v)\, T_{in}}{\rho_0\,\mathcal{V}\, C_v} = 3.2 \text{ C/s}$$

where $\rho_0 = p_0/RT_0 = 200/(0.287 \times 293) = 2.3783$ kg/m^3 is the initial density in the tank.

Problem 3.144

Step 1: The control volume is as shown in Figure.

Step 2: The fluid is incompressible and steady. Assume uniform properties at sections 1, 2 and 3. The flow area A_1 is much larger than A_2 so that $V_1 \simeq 0$. The friction head losses are between 1 and 2. The losses between 2 and 3 are negligible.

Step 3: Conservation of mass between 1 and 2 leads to

$$V_1 \simeq 0$$

Conservation of energy:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right) d\mathcal{V} + \int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} =$$
$$\dot{Q}_{\text{gained}} - \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

In order to determine the exit angle θ that maximizes the horizontal distance so that $z_3 - z_2$ is constant is obtained by applying Newton's second law for a small fluid element of volume $\delta\mathcal{V}$ and fixed mass $\rho\,\delta\mathcal{V}$. Noting that the only force acting on this element is gravity, then

$$\frac{d}{dt}(\rho\,\delta\mathcal{V}\,\mathbf{u}) = \rho\,\delta\mathcal{V}\,\mathbf{g}$$
$$\frac{d\mathbf{u}}{dt} = \mathbf{g}$$

The horizontal and vertical components are

$$\frac{dw}{dt} = -g \Rightarrow w - w_0 = -gt \Rightarrow w - V_2\,\sin\theta = -gt$$
$$\frac{du}{dt} = 0 \Rightarrow u = u_0 = V_2\,\cos\theta \Rightarrow x = V_2\,\cos\theta\, t$$

At 3 $w = 0$, $x = R/2$ where R is the horizontal range to be maximized. At 3, eliminating t from the above two equations leads to

$$R = 2\,V_2\,\cos\theta\,\frac{V_2\,\sin\theta}{g} = \frac{V_2^2\,\sin 2\theta}{g}$$

It is clear that R is maximum when $\theta = \pi/4$.

To determine V_2, we first apply the conservation of energy between 2 and 3:

$$\dot{m}\left(\frac{V_3^2}{2} + gz_3\right) - \dot{m}\left(\frac{V_2^2}{2} + gz_2\right) = 0$$
$$\Rightarrow\quad V_2^2 - V_3^2 = 2\,g\,(z_3 - z_2)$$

To relate V_3 to V_2, we apply conservation of momentum in the x direction for the control volume consisting of the jet 2-3, we get

$$\dot{m}\,V_2\,\cos\theta = \dot{m}V_3$$

Substituting in the energy equation, we get

$$\begin{aligned} \dot{m}\left(\frac{V_3^2}{2}+gz_3\right)-\dot{m}\left(\frac{V_2^2}{2}+gz_2\right) &= 0 \\ \Rightarrow \quad V_2^2\left(1-\cos^2\theta\right) &= 2\,g\,(z_3-z_2) \\ \Rightarrow \quad V_2^2\,\sin^2\theta &= 2\,g\,(z_3-z_2) \end{aligned}$$

With $\theta=\pi/4$, we get

$$V_2^2 = 4\,g\,(z_3-z_2)$$

With $z_3-z_2=25$ m, we get $V_2=31.32$ m/s.

To determine the power delivered to the pump, we apply conservation of energy between 1 and 2

$$\begin{aligned} \dot{m}\left(\hat{h}_2+\frac{V_2^2}{2}+gz_2\right)-\dot{m}\left(\hat{h}_1+\frac{V_1^2}{2}+gz_1\right) &= \dot{W}_{pump}-\dot{m}\,g\,h_f \\ \Rightarrow \quad \dot{m}\left(C_vT_2+\frac{p_2}{\rho_2}+\frac{V_2^2}{2}+gz_2\right)-\dot{m}\left(C_vT_1+\frac{p_1}{\rho_1}+\frac{V_1^2}{2}+gz_1\right) &= \dot{W}_{pump}-\dot{m}\,g\,h_f \end{aligned}$$

The flow is assumed to be isothermal $T_1=T_2$, also $p_1\simeq p_2=p_a$, $V_1\simeq 0$, then

$$\dot{W}_{pump}=\rho\,V_2\,\frac{\pi D_2^2}{2}\,\left[\frac{V_2^2}{2}+g\,(z_2-z_1+h_f)\right]=26.245\text{ kW}$$