MECH 314 Introduction to Fluid Mechanics
American University of Beirut, Spring 2009

November 18, 2009
Handout # PS3

Problem Set 3 Solution

Textbook:

Conservation of mass:
Problems 3.19, 3.23, 3.25, 3.29

Conservation of momentum:
Problems 3.39, 3.53, 3.80, 3.82

Conservation of energy:
Problems 3.132, 3.137, 3.143, 3.144

Problem 3.19

Step 1: The control volume is as shown in Figure.

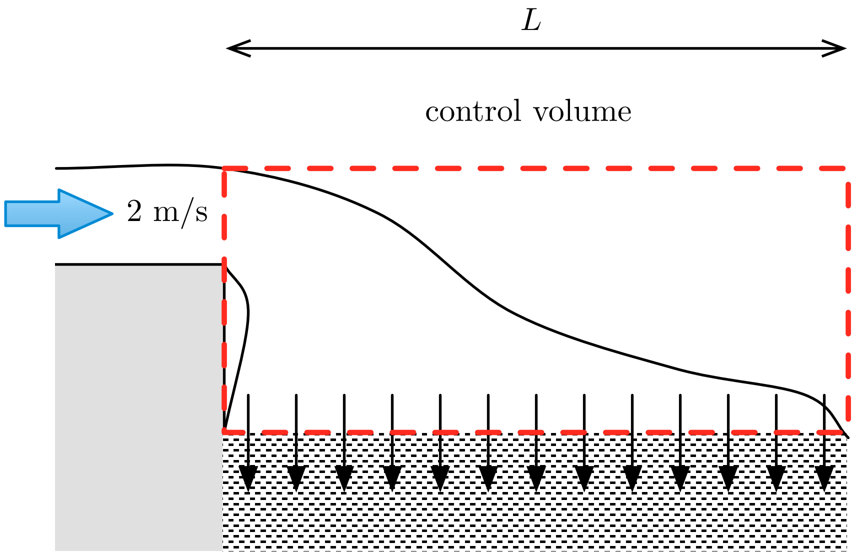

Figure 1: Control Volume Selection.

Step 2: The fluid is incompressible, i.e. constant density. The flow is steady.

Step 3: Apply the conservation of mass:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0 \\
\Rightarrow \quad & 0 + \rho \, u_{out} \, A_{out} - \rho \, u_{in} \, A_{in} = 0 \\
\Rightarrow \quad & u_{out} \, L \, W = u_{in} \, H \, W \\
\Rightarrow \quad & L = H \frac{u_{in}}{u_{out}}
\end{aligned}$$

With $H = 20$ cm, $u_{in} = 2$ m/s, and $u_{out} = 8$ mm/s, then $L = 50$ m.

Problem 3.23

Step 1: The control volume is as shown in Figure. The left side of the control surface is moving with the serum.

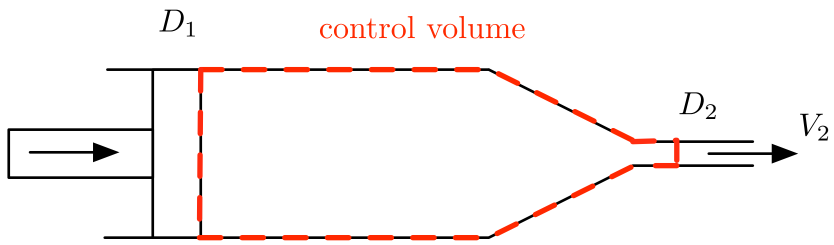

Figure 2: Control Volume Selection.

Step 2: The fluid is incompressible; i.e. constant density. The flow is unsteady so that we have to take care of the rate of change of mass inside the control volume.

Step 3: Apply the conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0$$

Notice that the velocity in the second term is the velocity of the fluid at the control surface relative to the control surface. With that in mind and noting the rate of change of volume is $\dot{\mathcal{V}} = d(A_1 \, x)/dt = A_1 dx/dt = -A_1 u_{plunger}$, then

$$\begin{aligned} & -\rho \, A_1 \, u_{plunger} + \rho \, u_{out,needle} \, A_{out,needle} + \rho \, u_{out,leakage} \, A_{out,leakage} = 0 \\ \Rightarrow \quad & -\rho \, A_1 \, u_{plunger} + \rho \, \dot{\mathcal{V}}_{serum} + \rho \, \dot{\mathcal{V}}_{leakage} = 0 \end{aligned}$$

where $\dot{\mathcal{V}}_{serum} = u_{out,needle} \, A_{out,needle}$.

(a) For the case of no leakage,

$$u_{plunger} = \frac{\dot{\mathcal{V}}_{serum}}{A_1} = \frac{\dot{\mathcal{V}}_{serum}}{\pi D_1^2/4}$$

With $\dot{\mathcal{V}}_{serum} = 6 \text{ cm}^3/\text{s} = 0.3661 \text{ in}^3/\text{s}$ and $A_1 = 0.4418 \text{ in}^2$, then $u_{plunger} = 0.8287$ in/s.

(b) For the case of leakage where $\dot{\mathcal{V}}_{leakage} = 0.1 \dot{\mathcal{V}}_{serum}$, then

$$u_{plunger} = \frac{1.1 \, \dot{\mathcal{V}}_{serum}}{A_1} = \frac{1.1 \, \dot{\mathcal{V}}_{serum}}{\pi D_1^2/4}$$

Then $u_{plunger} = 0.9115$ in/s.

Problem 3.25

Step 1: The control volume is as shown in Figure. The upper side of the control surface extends to the limit $H \to \infty$.

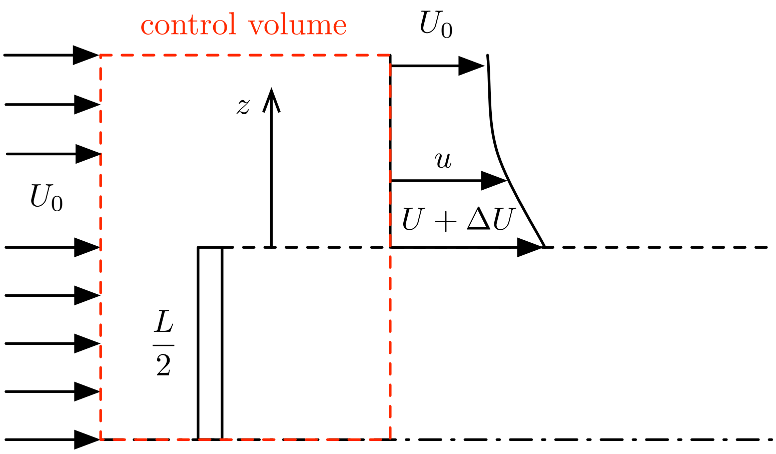

Figure 3: Control Volume Selection.

Step 2: The fluid is incompressible; i.e. constant density. The flow is steady.

Step 3: Apply the conservation of mass:

$$
\begin{aligned}
& \frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0 \\
\Rightarrow \quad & 0 + \int_{L/2}^{H} \rho \, u \, b \, dz - \int_{0}^{H} \rho \, U_0 \, b \, dz = 0 \\
\Rightarrow \quad & 0 + \int_{L/2}^{H} \rho \, (u - U_0) \, b \, dz - \int_{0}^{L/2} \rho \, U_0 \, b \, dz = 0 \\
\Rightarrow \quad & \int_{L/2}^{H} (u - U_0) \, dz = U_0 \frac{L}{2}
\end{aligned}
$$

With $u \simeq U_0 + \Delta U \, e^{-z/L}$, then

$$
\begin{aligned}
& \lim_{H \to \infty} \int_{L/2}^{H} \Delta U \, e^{-z/L} \, dz = U_0 \frac{L}{2} \\
\Rightarrow \quad & \Delta U \, L \, e^{-\frac{1}{2}} = U_0 \frac{L}{2} \\
\Rightarrow \quad & \Delta U = U_0 \frac{e^{\frac{1}{2}}}{2}
\end{aligned}
$$

Problem 3.29

Step 1: The control volume is as shown in Figure.

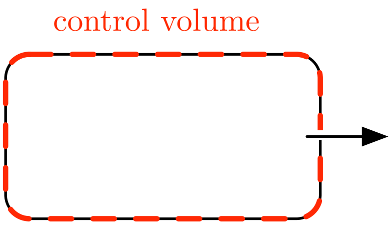

Figure 4: Control Volume Selection.

Step 2: The fluid is compressible and unsteady. The control volume has a constant volume $\mathcal{V}$, but the density is variable in time but not in space so that $\rho = \rho(t)$.

Step 3: Apply the conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0$$

Since the density is a function of time only, then

$$\begin{aligned} \mathcal{V}\frac{d\rho}{dt} + \dot{m}_{out} &= 0 \\ \Rightarrow \quad \mathcal{V}\frac{d\rho}{dt} + C\,\rho &= 0 \\ \Rightarrow \quad \frac{d\rho}{\rho} &= -\frac{C}{\mathcal{V}} \end{aligned}$$

Integrating from $t = 0$ to t, then

$$\begin{aligned} \ln\left(\frac{\rho}{\rho_0}\right) &= -\frac{C}{\mathcal{V}}\,t \\ \Rightarrow \quad \rho(t) &= \rho_0\, e^{-\frac{C}{\mathcal{V}}t} \end{aligned}$$

Ideal gas law gives $\rho_0 = \frac{p_0}{RT_0}$. Also $\dot{m}_{out,0} = C\rho_0$ so that $C = \frac{\dot{m}_{out,0}}{\rho_0}$. The tank volume is $\mathcal{V} = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$. The time Δt it takes for the density to drop by a factor of 2 is

$$\begin{aligned} \ln\left(\frac{\frac{1}{2}\rho_0}{\rho_0}\right) &= -\frac{C}{\mathcal{V}}\,\Delta t \\ \Rightarrow \quad \Delta t &= -\frac{\frac{4}{3}\pi\left(\frac{D}{2}\right)^3}{\frac{\dot{m}_{out,0}}{\rho_0}} \ln\left(\frac{1}{2}\right) \\ \Rightarrow \quad \Delta t &= -\frac{\frac{4}{3}\pi\left(\frac{D}{2}\right)^3}{\frac{\dot{m}_{out,0}\,RT_0}{p_0}} \ln\left(\frac{1}{2}\right) \end{aligned}$$

Problem 3.39

Step 1: The control volume is as shown in Figure.

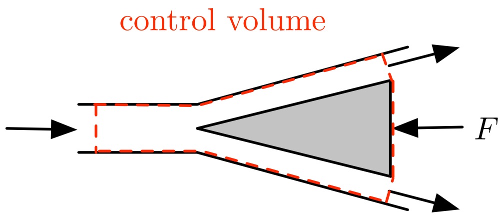

Figure 5: Control Volume Selection.

Step 2: The fluid is incompressible and steady.

Step 3: Apply the conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0$$

Leading to $\dot{m}_2 = \dot{m}_3 = \frac{1}{2}\dot{m}_1$.

Apply the conservation of momentum:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, \mathbf{u} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u} \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \sum \mathbf{F}_{ext}$$

Taking the x component of the above equation leads to

$$0 + u_2 \, \dot{m}_2 \, \cos\frac{\theta}{2} + u_3 \, \dot{m}_3 \, \cos\frac{\theta}{2} - u_1 \, \dot{m}_1 = -F$$

With $u_1 = u_2 = u_3$ and $\dot{m}_2 = \dot{m}_3 = \frac{1}{2}\dot{m}_1$ we get

$$\cos\frac{\theta}{2} = \frac{u_1 \, \dot{m}_1 - F}{u_1 \dot{m}_1} = 1 - \frac{F}{u_1 \dot{m}_1}$$

With u_1 =6 m/s and $\dot{m}_1 = \rho \, u_1 \, h_1 = 240$ kg/s/m. Then $\theta = 81$ deg.

Problem 3.53

Step 1: The control volume is as shown in Figure. The forces applied on the control volume are due to pressure and viscous drag (friction at the wall.)

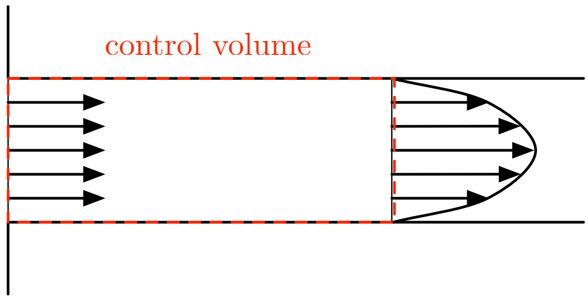

Figure 6: Control Volume Selection.

Step 2: The fluid is incompressible and steady.

Step 3: Apply the conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0$$

Noting that the unsteady term is zero and $d\mathcal{S} = 2\pi \, r \, dr$, we get the following results

- For Laminar flow: $u_{max} = 2\,U_0$. (Please derive this result.)
- For Turbulent flow: $u_{max} = \frac{60}{49}\,U_0$. (Please derive this result.)

Apply the conservation of momentum:

$$\begin{aligned} & \frac{d}{dt}\int_{\mathcal{V}} \rho \, \mathbf{u} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u} \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \sum \mathbf{F} \\ \Rightarrow \quad & \frac{d}{dt}\int_{\mathcal{V}} \rho \, \mathbf{u} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \, \mathbf{u} \, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = -\int_{\mathcal{S}} p_{gage} \, \hat{\mathbf{n}} \, d\mathcal{S} + \mathbf{F}_{drag} \end{aligned}$$

Taking the x component of the momentum equation

$$0 + \rho \int_0^R u_2^2 \, 2\pi \, r \, dr - \rho \int_0^R U_0^2 \, 2\pi \, r \, dr = -(p_2 - p_1)\,\pi \, R^2 - F_{drag}$$

Then

- For Laminar flow:

$$F_{drag} = \pi R^2 \left[(p_1 - p_2) - \frac{\rho U_0^2}{3} \right]$$

- For Turbulent flow:

$$F_{drag} = \pi R^2 \left[(p_1 - p_2) - \frac{\rho U_0^2}{49} \right]$$

Turbulent flows experience larger viscous drag at the wall.

Problem 3.80

Step 1: The control volume is as shown in Figure. The forces applied on the control volume are due to pressure which is assumed to be hydrostatic. Viscous forces at the river bottom are neglected.

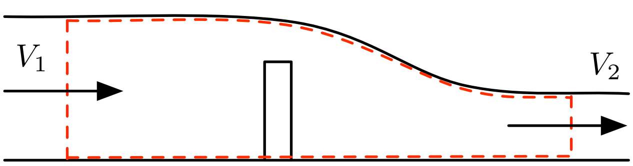

Figure 7: Control Volume Selection.

Step 2: The fluid is incompressible and steady.

Step 3: Apply the conservation of mass:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\mathcal{V} + \int_{\mathcal{S}} \rho\, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = 0$$

Noting that the unsteady term is zero then $V_2 = V_1 \frac{h_1}{h_2}$.

Apply the conservation of momentum:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, \mathbf{u}\, d\mathcal{V} + \int_{\mathcal{S}} \rho\, \mathbf{u}\, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = \sum \mathbf{F}$$

$$\Rightarrow \quad \frac{d}{dt}\int_{\mathcal{V}} \rho\, \mathbf{u}\, d\mathcal{V} + \int_{\mathcal{S}} \rho\, \mathbf{u}\, \mathbf{u}_{rel} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = -\int_{\mathcal{S}} p_{gage}\, \hat{\mathbf{n}}\, d\mathcal{S} + \mathbf{F}_{viscous} + \mathbf{F}_{obstacle}$$

Noting that the flow is steady and the viscous drag is neglected and that the velocity at sections 1 and 2 is uniform. Taking the x component of the momentum equation

$$0 + \dot{m}\, V_2 - \dot{m}\, V_1 = -\int_0^{h_2} p_{2,gage}\, b\, dz + \int_0^{h_1} p_{1,gage}\, b\, dz - F_{obstacle}$$

Note that if the flow exerts a force $F_{obstacle}$ on the obstacle, then the obstacle exerts a force of $-F_{obstacle}$ on the fluid. Noting that the hydrostatic pressure is $p_{1,gage} = \rho\, g\, (h_1 - z)$ and $p_{2,gage} = \rho\, g\, (h_2 - z)$, then

$$\begin{aligned} F_{obstacle} &= \dot{m}\,(V_1 - V_2) + \frac{\rho\, g\, b}{2}\left(h_1^2 - h_2^2\right) \\ &= \rho\, V_1\, b\, h_1 \left(V_1 - V_1 \frac{h_1}{h_2}\right) + \frac{\rho\, g\, b}{2}\left(h_1^2 - h_2^2\right) \\ &= -\rho\, V_1^2\, b\, h_1 \left(\frac{h_1}{h_2} - 1\right) + \frac{\rho\, g\, b}{2}\left(h_1^2 - h_2^2\right) \end{aligned}$$

Problem 3.82

Step 1: The control volume, shown in Figure, includes the car and the inlet and outlet jets. The reference frame is attached to the accelerating car and is therefore non-inertial.

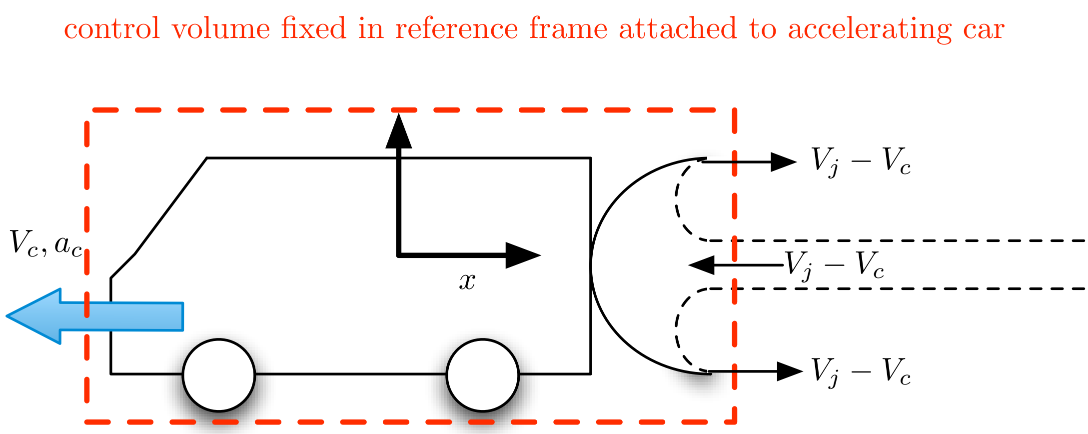

Figure 8: Control Volume Selection.

Step 2: The fluid is incompressible. In the chosen reference frame, the flow is steady.

Step 3: Apply the conservation of mass for the jet in the accelerating reference frame.

$$\rho(V_j - V_c)A_j = \rho V_e A_e$$

where $A_j = \pi D_j^2/4$ and V_e and A_e are respectively the jet exit velocity in the accelerating reference frame and flow area. V_e may be determined by applying Bernoulli's equation along a stream in the accelerating reference frame, assuming the jet flow to be inviscid, steady and of constant density. Then

$$p_1 + \frac{1}{2}\rho(V_j - V_c)^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_e^2 + \rho g z_2$$

Neglecting effects of gravity (this means that $\rho g(z_2 - z_1) << \frac{1}{2}\rho(V_j - V_c)^2$) and noting that $p_1 = p_2 = p_a$, we conclude that $V_e = (V_j - V_c)$ and consequently $A_e = A_j$.

Next we apply conservation of momentum in an accelerating reference frame, and considering the x component only

$$\sum F_x - m_c a_c = \frac{\partial}{\partial t}\int_{CV} \rho u_x \, d\mathcal{V} + \int_{CS} \rho u_x (\mathbf{u}_r \cdot \hat{\mathbf{n}}) \, d\mathcal{S}$$

where m_c and a_c are respectively the mass and acceleration of the car, u_x is the $x-$ component of the velocity in the accelerating reference frame, and $\mathbf{u}_r$ is the velocity of the flow with respect to the control surface.

Since the flow is steady in the car reference frame, and the forces due to air drag and wheel friction with ground are neglected, then

$$-m_c a_c = \int_{CS} \rho u_x (\mathbf{u}_r \cdot \hat{\mathbf{n}})\, d\mathcal{S}$$

Noting that $\int_{CS} \rho(\mathbf{u}_r \cdot \hat{\mathbf{n}})\, d\mathcal{S} = -\rho(V_j - V_c)A_j$ for the inlet flow and $\rho(V_j - V_c)A_j$ for the outlet flow, where $V_j - V_c$ is the speed of the jet relative to the car moving at V_c. And noting that $u_x = -(V_j - V_c)$ for the inlet flow and $(V_j - V_c)$ for the outlet flow, then

$$-m_c a_c = 2\rho(V_j - V_c)^2 A_j$$

It is important to note the pressure is atmospheric throughout the control surface. In particular, the pressure across a free straight jet characterized by large Reynolds number $Re = \rho V_j D_j/\mu >> 1$ is atmospheric. This, along with Bernoulli's equation, will be explained and discussed further when we study inviscid flows.

Another important thing to look at is applying the conservation of mass for the jet by taking a fixed control mass with its right side fixed at the jet exit from the nozzle and its left side moving with the car. How will the conservation of mass look like? Can we neglect the transient term in this case?