Problem Set 4 Solution: Viscous Flow

"Advanced Fluid Mechanics Problems" by Shapiro and Sonin
Problems 6.1, 6.3, 6.7, 6.10, 6.16, 6.20, 6.21.

Problem 6.1

Refer to Figure 1 for the schematic. The flow is steady.

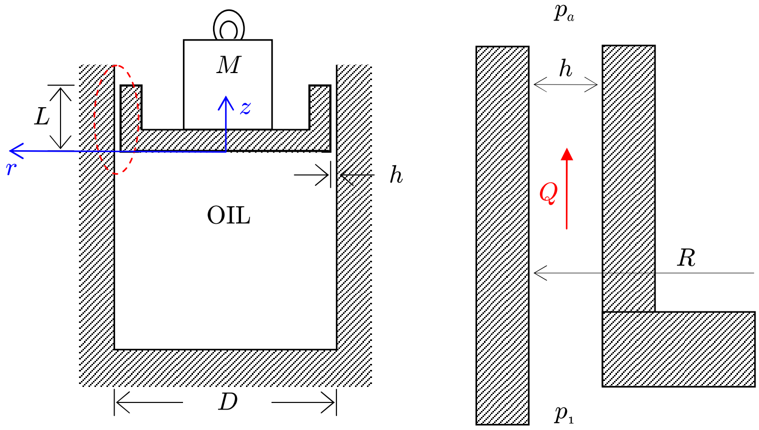

Figure 1: Schematic of Problem 6.1

(a) We assume that the flow is steady and fully developed in the channel between the piston and cylinder. The $z-$component of the momentum equation is

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

where gravity is neglected. We can neglect the term $\frac{1}{r}\frac{\partial u}{\partial r}$ because

$$\frac{\frac{1}{r}\frac{\partial u}{\partial r}}{\frac{\partial^2 u}{\partial r^2}} \simeq \frac{h}{R} << 1$$

where $R = D/2$. From the momentum conservation in the $r-$direction, we can show that

$$\frac{\partial p}{\partial r} \simeq 0 \Rightarrow p = p(z) \text{ and } \frac{\partial p}{\partial z} = \frac{dp}{dz}$$

Therefore equation (1) reduces to

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{\mu}\frac{dp}{dz}$$

subject to the boundary conditions

$$u = 0 \text{ at } r = R - h$$
$$u = 0 \text{ at } r = R$$

The velocity is then given by

$$u(r) = \frac{1}{2\mu}\frac{dp}{dz}\left(r^2 - (2R-h)r + R(R-h)\right)$$

The volume flow rate is then given by

$$\begin{aligned} Q &= 2\pi\left(R - \frac{h}{2}\right)\int_{R-h}^{h} u(r)dr \\ &= -2\pi\left(R - \frac{h}{2}\right)\frac{h^3}{12\mu}\frac{dp}{dz} \end{aligned}$$

We conclude from the last equation that dp/dz is constant,

$$\frac{dp}{dz} = -\frac{p_1 - p_a}{L} = -\frac{Mg}{\pi R^2 L}$$

where the shear force on the piston acting vertically upwards has been neglected. The leakage volume flow rate is finally given as

$$Q = \frac{Mgh^3}{6\mu R^2 L}\left(R - \frac{h}{2}\right) \simeq \frac{Mgh^3}{6\mu R L}$$

Problem 6.3

Refer to Figure 2 for the schematic. The flow is steady.

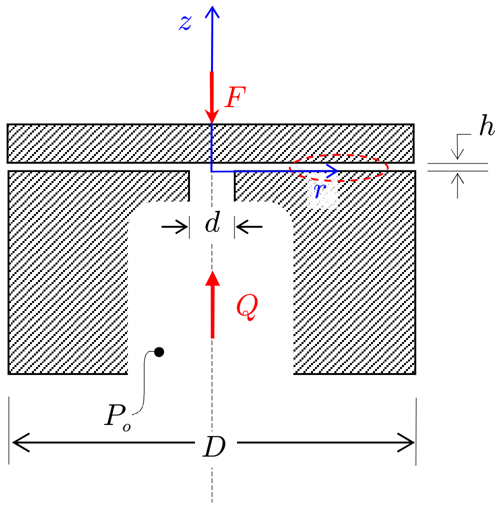

Figure 2: Schematic of Problem 6.3

(a) We assume that the flow is steady and fully developed in the channel between the upper and lowe plates away from the center, i.e. for $r > d/2$. The momentum equation in the $r-$direction is

$$\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

The ratio of the dominant inertia term to the dominant diffusion term is

$$\frac{\rho u\frac{\partial u}{\partial r}}{\mu\frac{\partial^2 u}{\partial z^2}}\simeq\frac{\rho\frac{\bar{u}^2}{r}}{\mu\frac{\bar{u}}{h^2}}=\frac{\rho\bar{u}h}{\mu}\frac{h}{r}=\mathrm{Re}_h\frac{h}{r}$$

(b) Noting that $Q = 2\pi r h\bar{u}$, the flow may be assumed to be inertia free if

$$\begin{aligned}
&\frac{\rho\bar{u}h}{\mu}\frac{h}{r}<<1\\
\Rightarrow\ &\frac{Q}{2\pi rh}\frac{\rho h}{\mu}\frac{h}{r}<<1\\
\Rightarrow\ &r^2>>\frac{\rho Qh}{2\pi\mu}\\
\Rightarrow\ &\sqrt{\frac{\rho Qh}{2\pi\mu}}<<r\leq R
\end{aligned}$$

(c) The inertia free flow is governed by

$$\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{dp}{dr}$$

subject to the boundary conditions

$$u = 0 \text{ at } z = 0$$
$$u = 0 \text{ at } z = h$$

The velocity is then given by

$$u = \frac{1}{2\mu}\frac{dp}{dr}\left(z^2 - zh\right)$$

The volume flow rate is

$$\begin{aligned} Q &= \frac{2\pi r}{2\mu}\frac{dp}{dr}\int_0^h \left(z^2 - zh\right) dz \\ &= -\frac{\pi r h^3}{6\mu}\frac{dp}{dr} \end{aligned}$$

The pressure distribution is therefore governed by

$$\begin{aligned} \frac{dp}{dr} &= -\frac{6\mu Q}{\pi h^3}\frac{1}{r} \\ \Rightarrow \quad p(r) - p_o &= -\frac{6\mu Q}{\pi h^3}\ln\left(\frac{2r}{d}\right) \end{aligned}$$

At $r = D/2$, the gage pressure is zero so that

$$Q = \frac{\pi p_o h^3}{6\mu \ln(D/d)}$$

The force is balance by the pressure distribution

$$\begin{aligned} F &= \int_{d/2}^{D/2} p(r)\, 2\pi r dr \\ &= 2\pi \int_{d/2}^{D/2} p_o \left[1 - \frac{\ln(2r/d)}{\ln(D/d)}\right] r\, dr \\ &= \frac{\pi D^2 p_o}{4}\left[\frac{1 - (d/D)^2}{2\ln(D/d)}\right] \end{aligned}$$

Problem 6.7

Refer to Figure 3 for the schematic. The flow is steady.

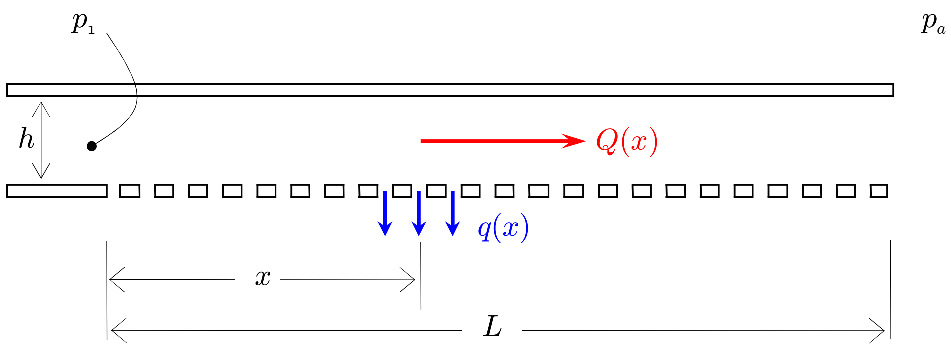

Figure 3: Schematic of Problem 6.7

(a) We apply the integral form of the conservation of mass for a control volume of height h ahd width dx,

$$Q + \frac{dQ}{dx}dx + qw\,dx - Q = 0$$
$$\Rightarrow \frac{dQ}{dx} = -wq = -kw\,[p(x) - p_a] \tag{1}$$

(b) The $x-$component of the momentum equation is

$$0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$$

subject to the conditions $u = 0$ at $y = 0$ and $y = h$. The solution is

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right)$$

leading to

$$Q = w\int_0^h u\,dy = -\frac{wh^3}{12\mu}\frac{dp}{dx}$$

Differentiating in x,

$$\frac{dQ}{dx} = -\frac{wh^3}{12\mu}\frac{d^2p}{dx^2} \tag{2}$$
$$\Rightarrow \frac{d^2p}{dx^2} = \frac{12\mu k}{h^2}\,[p(x) - p_a] \tag{3}$$

subject to the conditions

$$p = p_1 \text{ at } x = 0$$
$$p = p_a \text{ at } x = L$$

(c) In this case we have a conventional Poiseuille flow. If q is very small, then k is very small. From equation (3)

$$\frac{d^2p}{dx^2} \simeq 0$$
$$\Rightarrow \quad \frac{dp}{dx} = \frac{p_a - p_1}{L}$$

Therefore the location at which the gage pressure reaches half the inlet gage pressure is $x^* = L/2$.

(d) In this case $Q(x = L) \simeq 0$. A first order approximation in this xase is to assume that Q varies linearly from $Q = Q_0$ at $x = 0$ to $Q = 0$ at $x = L$,

$$\frac{dQ}{dx} \simeq -\frac{Q_0}{L}$$
$$\Rightarrow \quad Q \simeq Q_0 \left(1 - \frac{x}{L}\right)$$

From equation (2),

$$\frac{wh^3}{12\mu}\frac{d^2p}{dx^2} = \frac{Q_0}{L}$$

Leading to the pressure distribution

$$p(x) = \frac{6\mu Q_0}{wh^3 L}x^2 + \left(\frac{p_a - p_1}{L} - \frac{6\mu Q_0}{wh^3}\right)x + p_1 \tag{4}$$

To get Q_0, we integrate equation (1),

$$0 - Q_0 = \int_0^L -kw\left[p(x) - p_a\right]\, dx$$
$$\Rightarrow \quad Q_0 = kw\int_0^L \left[\frac{6\mu Q_0}{wh^3 L}x^2 + \left(\frac{p_a - p_1}{L} - \frac{6\mu Q_0}{wh^3}\right)x + p_1 - p_a\right] dx$$
$$\Rightarrow \quad Q_0 = \frac{wh^3(p_1 - p_a)}{2\mu L + 2h^3/kL} \simeq \frac{wh^3(p_1 - p_a)}{2\mu L}$$

Note $kL >> h^3/\mu L$ in this case. Substituting Q_0 in equation (4), we get

$$\frac{p(x) - p_a}{p_1 - p_a} = 3\frac{x^2}{L^2} - 4\frac{x}{L} + 1$$

Now we find x^* such that $(p(x^*) - p_a)/(p_1 - p_a) = \frac{1}{2}$, resulting in $x^* = 0.14L$.

(e) The differential equation

$$\frac{d^2p^*}{dx^2} = \alpha p^*(x)$$

where $p^*(x) = \frac{p(x)-p_a}{p_1-p_a}$ and $\alpha = \frac{12\mu k}{h^2}$ have the general solution

$$p^*(x) = C_1 e^{x\sqrt{\alpha}} + C_2 e^{-x\sqrt{\alpha}}$$

subject to the boundary conditions $p^* = 1$ at $x = 0$ and $p^* = 0$ at $x = L$, we get

$$C_1 + C_2 = 1$$
$$C_1 e^{L\sqrt{\alpha}} + C_2 e^{-L\sqrt{\alpha}} = 0$$

leading to

$$\frac{p(x) - p_a}{p_1 - p_a} = \frac{e^{(L-x)\sqrt{\alpha}} - e^{-(L-x)\sqrt{\alpha}}}{e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}}$$

- case 1: in the limit $k \to 0$, i.e. $\alpha = 0$,

$$\frac{p(x) - p_a}{p_1 - p_a} = \frac{L - x}{L}$$

  at $x = L/2$, we get $\frac{p(x)-p_a}{p_1-p_a} = 1/2$.

- case 2: in the limit $L\sqrt{\alpha} >> 0$, ,

$$\frac{p(x) - p_a}{p_1 - p_a} = e^{-x\sqrt{\alpha}}$$

  at $x = \frac{\ln(2)}{\sqrt{\alpha}}$, we get $\frac{p(x)-p_a}{p_1-p_a} = 1/2$.

Problem 6.10

Refer to Figure 4 for the schematic.

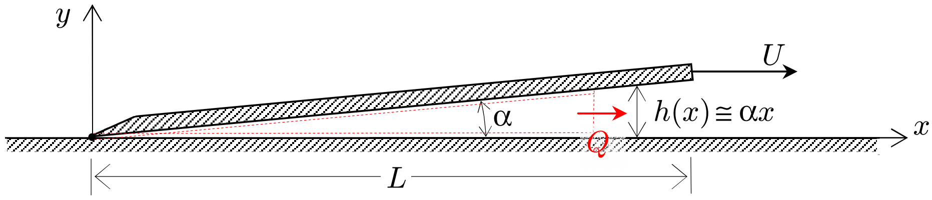

Figure 4: Schematic of Problem 6.10

(a) the ratio of the inertia to viscous forces is

$$\frac{\rho u \frac{\partial u}{\partial x}}{\mu \frac{\partial^2 u}{\partial y^2}} \sim \frac{\rho \frac{U^2}{x}}{\mu \frac{U}{\alpha^2 x^2}} = \frac{\rho U x}{\mu}\alpha^2 = \mathrm{Re}_x\, \alpha^2$$

Therefore the criteria for modeling the flow as inertia free is to have $\mathrm{Re}_x\, \alpha^2 << 1$.

(b) Locally Couette flow. The governing equation is

$$0 = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$u = 0 \text{ at } y = 0$$
$$u = U \text{ at } y = h$$

The velocity is then given by

$$u = -\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yh\right) + U\frac{y}{h}$$

The volume flow rate is given by

$$\begin{aligned} Q &= \int_0^h u dy = \frac{h^3}{12\mu}\frac{dp}{dx} + \frac{Uh}{2} \\ &= \frac{\alpha^3 x^3}{12\mu}\frac{dp}{dx} + \frac{U\alpha x}{2} \end{aligned}$$

By applying the integral form of the conservation of mass for the control volume shown in Figure 4, we get[1]

$$Q = 0$$

Then

$$\begin{aligned} \frac{dp}{dx} &== -\frac{6\mu U}{\alpha^2}\frac{1}{x^2} \\ \Rightarrow\ p - p_L &= \frac{6\mu U}{\alpha^2}\frac{L-x}{xL} \end{aligned}$$

[1]It may be easier to get the result in a reference frame moving with the blade.

Problem 6.16

Refer to Figure 5 for the schematic.

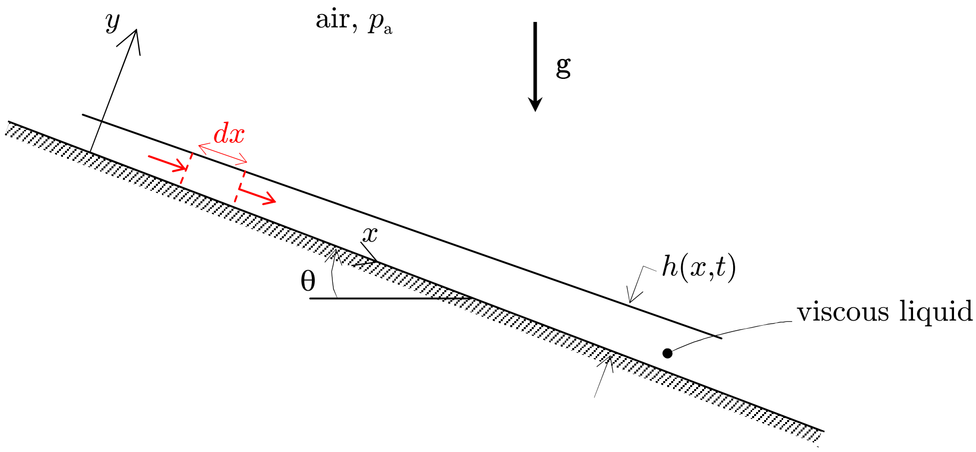

Figure 5: Schematic of Problem 6.16

(a) We apply conservation of mass for a control volume shown in the figure,

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\mathbf{u}\cdot\hat{\mathbf{n}}\, d\mathcal{S} = 0$$
$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho h\, dx) + \rho\frac{dQ}{dx}dx = 0$$
$$\Rightarrow \quad \frac{\partial h}{\partial t} + \frac{dQ}{dx} = 0 \tag{5}$$

where Q is the volume flowrate. The differential form of the momentum conservation in the $x-$direction is

$$0 = \mu\frac{\partial u^2}{\partial y^2} + \rho g\,\sin\theta$$

where the pressure gradient is zero and the inertia term is neglected. The boundary conditions are

$$u = 0 \text{ at } y = 0$$
$$\tau = \mu\frac{\partial u}{\partial y} = 0 \text{ at } y = h$$

The velocity is then given by

$$u = \frac{\rho g\,\sin\theta}{\mu}\left(hy - \frac{y^2}{2}\right)$$

The volume flow rate is

$$Q = \int_0^h u\, dy = \frac{\rho g\,\sin\theta}{3\mu}h^3$$
$$\Rightarrow \quad \frac{dQ}{dx} = \frac{\rho g\,\sin\theta}{\mu}h^2\frac{dh}{dx}$$

Combining with equation (5), we get

$$\frac{\partial h}{\partial t} + \frac{\rho g\ \sin\theta}{\mu} h^2 \frac{dh}{dx} = 0$$

(b)