Problem Set 2 Solution: Inviscid Flow

"Advanced Fluid Mechanics Problems" by Shapiro and Sonin
Problems 4.4, 4.7, 4.8, 4.9, 4.10, 4.13, 4.18, 4.19, 4.21, 4.23, 4.24, 4.28.

Problem 4.4

Refer to Figure 1 for the schematic. The flow is 1D, inviscid and incompressible. Gravitational effects are negligible.

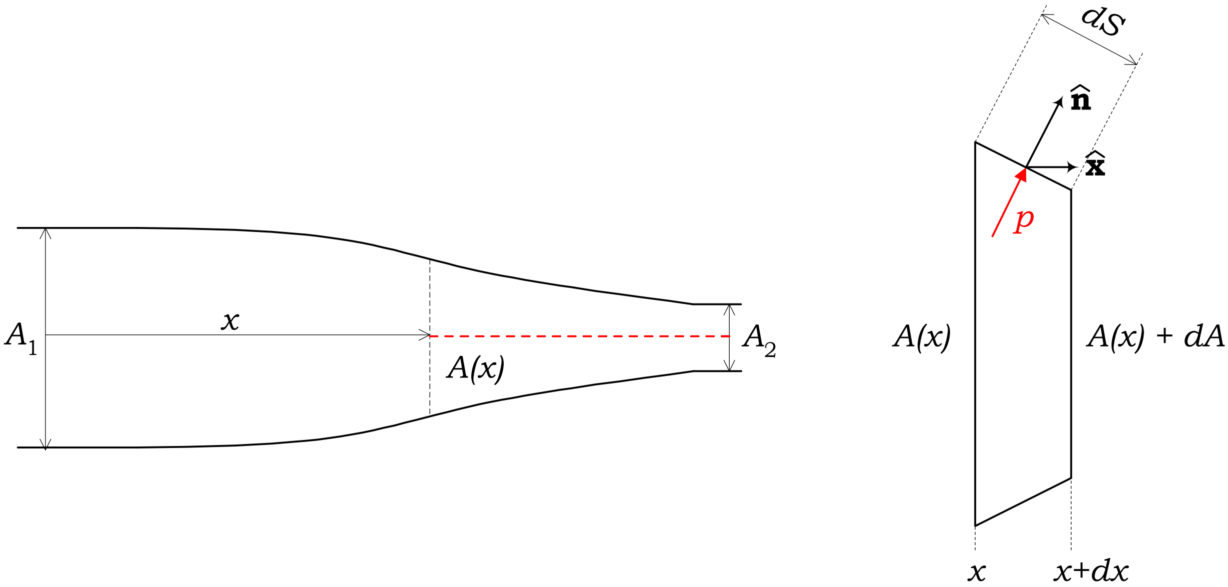

Figure 1: Schematic of Problem 4.4

(a) Applying Bernoulli's equation along a streamline from location x to location 2,

$$\begin{aligned}
\frac{p_2}{\rho} + \frac{u_2^2}{2} &= \frac{p(x)}{\rho} + \frac{u^2(x)}{2} \\
\Rightarrow p(x) - p(a) &= \frac{\rho}{2}\left(u_2^2 - u^2(x)\right) \\
\Rightarrow p(x) - p(a) &= \frac{\rho Q^2}{2}\left(\frac{1}{A_2^2} - \frac{1}{A^2(x)}\right)
\end{aligned}$$

(b) We consider a differential volume between x and $x + dx$. The cross sectional areas are respectively $A(x)$ and $A(x) + (dA/dx)dx$. The nozzle wall area onto which the pressure acts is dS so that the corresponding force $x-$component is $dF_x = p(x)dS\,\hat{\mathbf{n}} \cdot \hat{\mathbf{x}} = p(x)\,dA$, where $\hat{\mathbf{n}}$ is the unit vector normal to dS and pointing into the wall. The total force component in the x direction is

$$\begin{aligned}
F_x &= \int_1^2 dF_x = \int_{A_1}^{A_2} p dA = \int_{A_1}^{A_2} \frac{\rho Q^2}{2}\left(\frac{1}{A_2^2} - \frac{1}{A^2(x)}\right) dA \\
\Rightarrow F_x &= \rho Q^2 \frac{(A_1 - A_2)^2}{2A_1 A_2^2}
\end{aligned}$$

(c) The product $p\,dA$ has always the same sign.

- If $A \nearrow$, $dA > 0 \Rightarrow u \searrow \Rightarrow p \nearrow$
- If $A \searrow$, $dA < 0 \Rightarrow u \nearrow \Rightarrow p \searrow$

Problem 4.7

Refer to Figure 2 for the schematic.

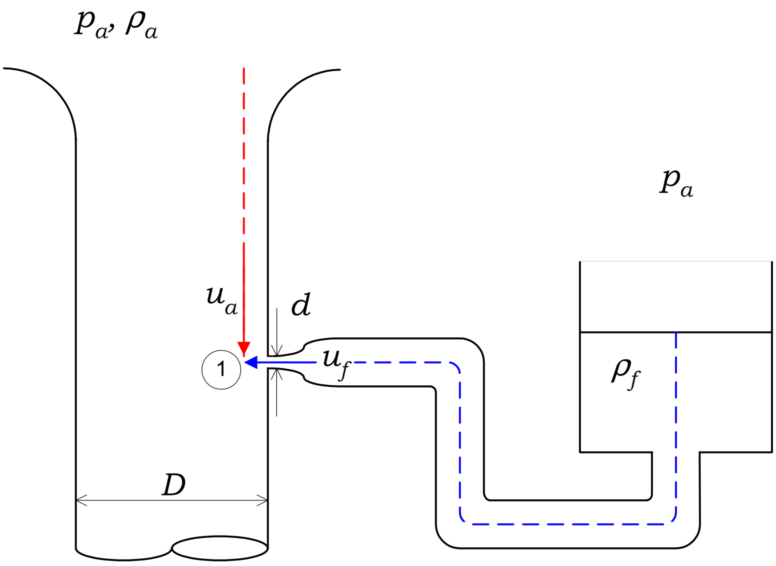

Figure 2: Schematic of Problem 4.7

The fuel-air mass flow rate ratio is

$$\alpha = \frac{\dot{m}_f}{\dot{m}_a} = \frac{\rho_f \, u_f \, (\pi d^2/4)}{\rho_a \, u_a \, (\pi D^2/4)}$$
$$\Rightarrow \frac{d}{D} = \left(\alpha \frac{\rho_a}{\rho_f} \frac{u_a}{u_f} \right)^{1/2}$$

In order to determine the ration u_a/u_f, we apply Bernoulli's equation along two stream lines, one for air from the ambient to location 1 in the vicinity of the fuel jet and one for fuel from the reservoir to location 1.

$$\text{for air} \qquad \frac{p_a}{\rho_a} = \frac{p_1}{\rho_a} + \frac{u_a^2}{2}$$
$$\text{for fuel} \qquad \frac{p_a}{\rho_f} = \frac{p_1}{\rho_f} + \frac{u_f^2}{2}$$

so that

$$\frac{u_a}{u_f} = \left(\frac{\rho_f}{\rho_a} \right)^{1/2}$$

Then

$$\frac{d}{D} = \alpha^{1/2} \left(\frac{\rho_a}{\rho_f} \right)^{1/4}$$

Problem 4.8
Refer to Figure 3 for the schematic. The flow is inviscid and incompressible.

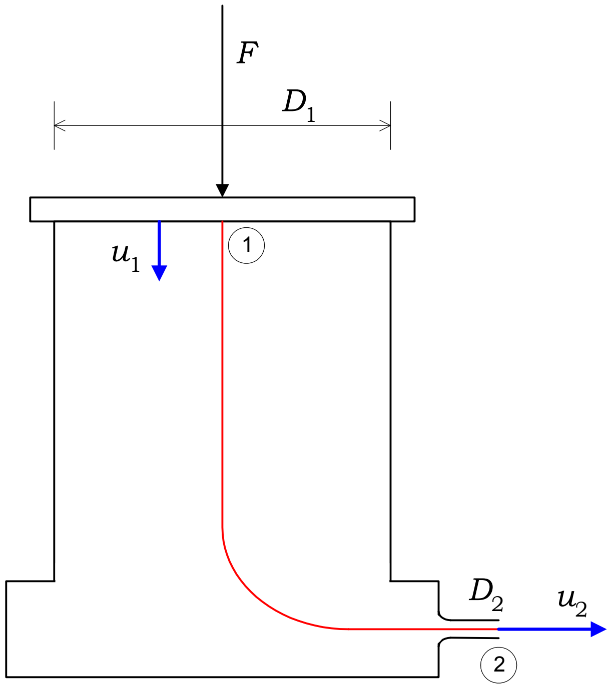

Figure 3: Schematic of Problem 4.8

(a) We apply Bernoulli's equation along a stream line starting from location 1 the top of the bellows and ending at location 2 at the nozzle exit.

$$\begin{aligned} \frac{p_1}{\rho} + \frac{u_1^2}{2} &= \frac{p_a}{\rho} + \frac{u_2^2}{\rho} \\ \Rightarrow p_1 - p_a &= \frac{\rho}{2}\left(u_2^2 - u_1^2\right) \\ \Rightarrow \frac{F}{A_1} &= \frac{\rho}{2}\left(u_2^2 - u_1^2\right) \end{aligned} \tag{1}$$

Next we apply the conservation of mass in integral form for a control volume that is moving with the piston

$$\begin{aligned} \frac{d}{dt}(\rho \mathcal{V}) + \rho\, u_2 \frac{\pi D_2^2}{4} = 0 \\ \Rightarrow u_2 = -\frac{4}{\pi D_2^2}\frac{d\mathcal{V}}{dt} \end{aligned}$$

By also applying conservation of mass in integral form for fixed control volume we find the relation

$$u_1\, D_1^2 = u_2\, D_2^2 \Rightarrow u_1 = -\left(\frac{D_2}{D_1}\right)^2 \frac{4}{\pi D_2^2}\frac{d\mathcal{V}}{dt}$$

Substituting expression for u_1 and u_2 into equation (1)

$$\frac{d\mathcal{V}}{dt} = -\frac{\pi D_2^2}{4}\left(\frac{2F}{\rho A_1}\frac{1}{1-\left(\frac{D_2}{D_1}\right)^4}\right)^{1/2}$$

Integrating from $t = 0$ to $t = \tau$ at which $\mathcal{V} = 0$,

$$\begin{aligned}\tau &= \mathcal{V}\frac{D_1}{D_2^2}\left(\frac{2\rho}{\pi F}\right)^{1/2}\left(1-\left(\frac{D_2}{D_1}\right)^4\right)^{1/2} \\ &\simeq \mathcal{V}\frac{D_1}{D_2^2}\left(\frac{2\rho}{\pi F}\right)^{1/2}\end{aligned}$$

(b) For STP air with $\mathcal{V} = \infty$ liter, $D_1 = 10$ cm, $D_2 = 1$ cm, and F =2kgf: $\tau = 0.2$ s.

Problem 4.9

Refer to Figure 4 for the schematic. The flow is inviscid and steady.

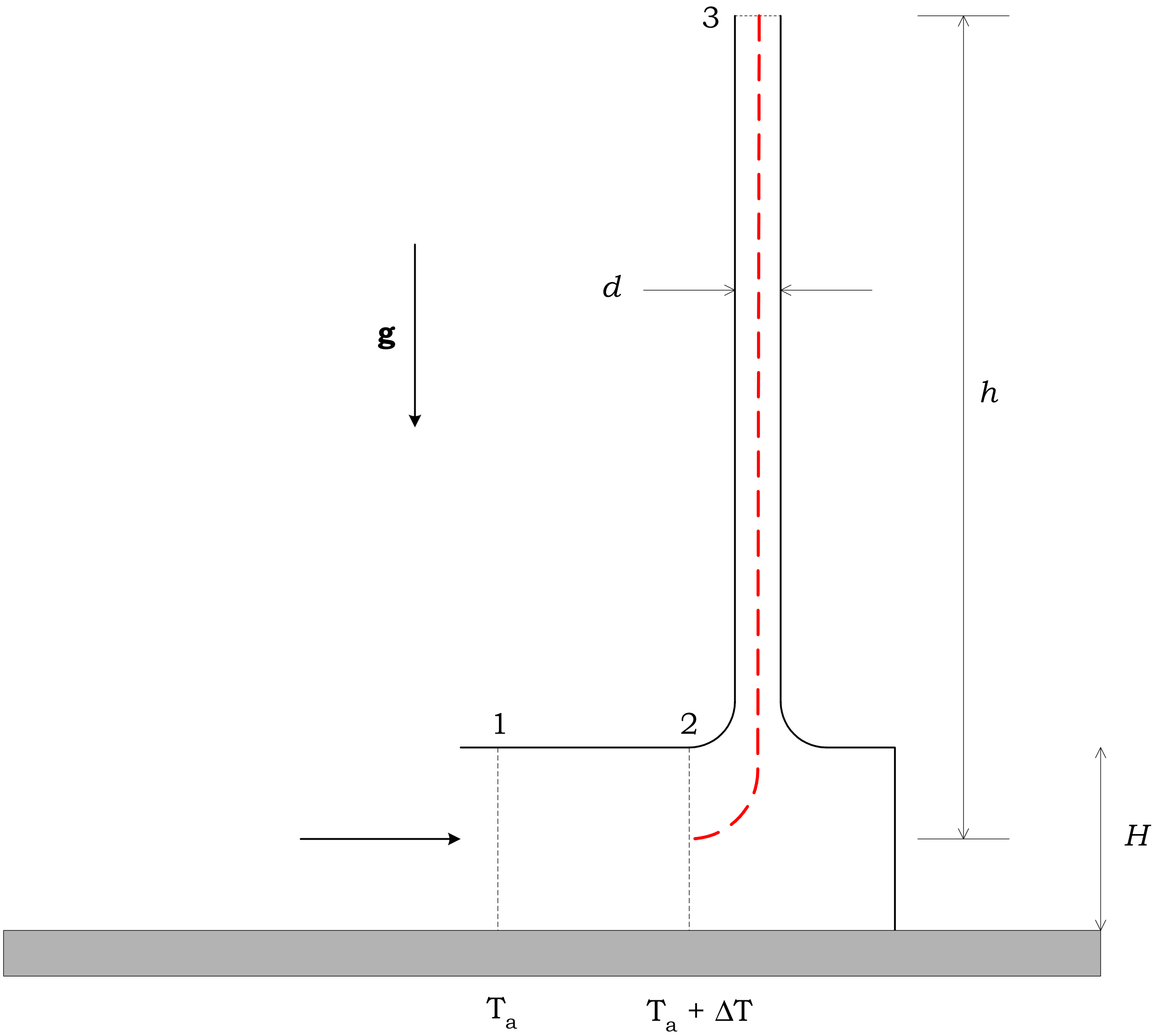

Figure 4: Schematic of Problem 4.9

(a) We apply Bernoulli's equation along the stream line $2-3$:

$$\int_2^3 dp + \int_2^3 \rho\, g\, z\, ds + \int_2^3 \frac{1}{2}\rho\, u^2\, ds = 0$$

$$\Rightarrow (p_3 - p_2) + \rho_H\, g\, h + \frac{1}{2}u_3^2 \simeq 0$$

$$\Rightarrow g\, h\, (\rho_a - \rho_H) \simeq \frac{1}{2}\rho_H\, u_3^2$$

$$\Rightarrow u_3 \simeq \left[2\, g\, h\left(\frac{\rho_a}{\rho_H} - 1\right)\right]^{1/2}$$

$$\Rightarrow u_3 \simeq \left[2\, g\, h\frac{\Delta T}{T_a}\right]^{1/2}$$

where $p_3 - p_2 = -\rho_a\, g\, h$, $\rho_2 = \rho_3 = \rho_H$, $u_2^2 << u_3^2$, and $p_a = \rho_a R T_a = \rho_H R\,(T_a + \Delta T)$.

(b) In the case the cap is closed,

$$
\begin{aligned}
\int_2^3 dp + \int_2^3 \rho\, g\, z\, ds + \int_2^3 \frac{1}{2}\rho\, u^2\, ds = 0 \\
\Rightarrow (p_3 - p_2) + \rho_H\, g\, h \simeq 0 \\
\Rightarrow p_3 \simeq p_a - \rho_H g\, h \\
\Rightarrow p_3 - p_{3a} \simeq g\, h\,(\rho_a - \rho_H) \\
\Rightarrow (\Delta p)_{cap} \simeq \rho_a\, g\, h \frac{\Delta T}{T_a + \Delta T}
\end{aligned}
$$

where $p_2 = p_a$, $p_{3a} = p_a - \rho_a\, g\, h$, $\rho_2 = \rho_3 = \rho_H$, $u_2 \simeq 0$, $u_3 = 0$ and $p_a = \rho_a R T_a = \rho_H R\,(T_a + \Delta T)$.

Problem 4.10

Refer to Figure 5 for the schematic. The flow is inviscid and quasi-steady.

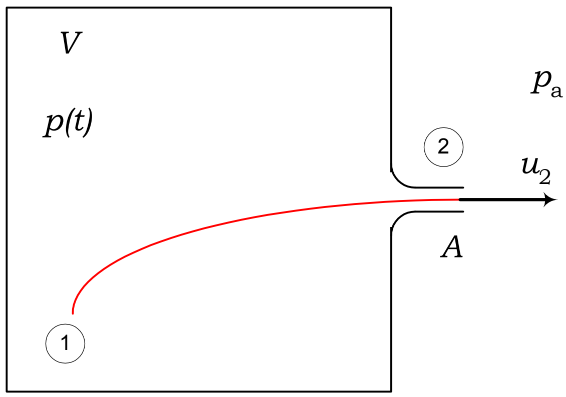

Figure 5: Schematic of Problem 4.10

We apply Euler's equation along a streamline from 1 inside the vessel to 2 at the exit

$$\int_1^2 \frac{dp}{\rho} + g(z_2 - z_1) + \frac{1}{2}\left(u_2^2 - u_1^2\right) + \int_1^2 \frac{\partial u}{\partial t} ds = 0$$

We neglect gravitational effects. The flow is quasi-steady so that $\int_1^2 \frac{\partial u}{\partial t} ds \simeq 0$. At location 1, $u_1 = 0$. Then

$$-2\int_1^2 \frac{dp}{\rho} = u_2^2 \tag{2}$$

Now we find an expression for u_2 by employing the integral form of the conservation of mass for a fixed control volume

$$\frac{d}{dt}(\rho\mathcal{V}) + \rho u_2 A = 0 \Rightarrow \mathcal{V}\frac{d\rho}{dt} + \rho u_2 A = 0 \Rightarrow u_2 = -\frac{\mathcal{V}}{A\,dt}\frac{d\rho}{\rho} \tag{3}$$

Substituting expression for u_2 (equation (3)) into equation (2)

$$\left(-2\int_1^2 \frac{dp}{\rho}\right)^{1/2} = -\frac{\mathcal{V}}{A\,dt}\frac{d\rho}{\rho} \tag{4}$$

(a) We employ $p = \rho R T_a$ and noting that $d\rho/\rho = dp/p$ then

$$\left(-2RT_a\int_1^2 \frac{dp}{p}\right)^{1/2} = -\frac{\mathcal{V}}{A\,dt}\frac{dp}{p} \Rightarrow \left(2RT_a \ln\frac{p_1}{p_2}\right)^{1/2} = -\frac{\mathcal{V}}{A\,dt}\frac{dp}{p}$$

$$\Rightarrow \left(2RT_a \ln\frac{p}{p_a}\right)^{1/2} = -\frac{\mathcal{V}}{A\,dt}\frac{dp}{p} \Rightarrow \frac{A}{\mathcal{V}}\left(2RT_a\right)^{1/2} dt = -\frac{1}{p\left(\ln\frac{p}{p_a}\right)^{1/2}}dp$$

where $p_2 = p_a$, $p_1 = p(t)$, $p'(t) = p(t) - p_a$. Integration from $t = 0$ to t and p from p_i to p, we get

$$\frac{A}{\mathcal{V}}\left(2RT_a\right)^{1/2} t = 2\left[\sqrt{\ln\frac{p_i}{p_a}} - \sqrt{\ln\frac{p}{p_a}}\right]$$

(b) We start with equation (4)

$$\left(-2\int_1^2 \frac{dp}{\rho}\right)^{1/2} = -\frac{\mathcal{V}}{A\,dt}\frac{d\rho}{\rho}$$

Noting that $p/p_i = (\rho/\rho_i)^\gamma \Rightarrow d\rho/\rho = (1/\gamma)dp/p$ then

$$\left(\frac{2\gamma}{\gamma-1}\frac{p_i}{\rho_i}\right)^{1/2}\left[\left(\frac{p}{p_i}\right)^{\frac{\gamma-1}{\gamma}} - \left(\frac{p_a}{p_i}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} = -\frac{\mathcal{V}}{\gamma\,A\,dt}\frac{dp}{p}$$

$$\Rightarrow -\frac{\gamma\,A}{\mathcal{V}}\left(\frac{2\gamma}{\gamma-1}\frac{p_i}{\rho_i}\right)^{1/2} dt = p^{-1}\left[\left(\frac{p}{p_i}\right)^{\frac{\gamma-1}{\gamma}} - \left(\frac{p_a}{p_i}\right)^{\frac{\gamma-1}{\gamma}}\right]^{-1/2} dp$$

Performing Taylor series expansion in p'/p_a around zero and integrating from $t=0$ to t yields

$$-\frac{\gamma\,A}{\mathcal{V}}\left(\frac{2\gamma}{\gamma-1}\frac{p_i}{\rho_i}\right)^{1/2} t = \frac{2}{(\alpha\,p_a)^{1/2}}\left({p'}^{1/2} - {p'_i}^{1/2}\right)$$

$$\Rightarrow p' = \left[{p'_i}^{1/2} - \left(\frac{p_a}{2}\right)^{1/2}\frac{\gamma\,A}{\mathcal{V}}\left(\frac{p_i}{\rho_i}\right)^{1/2} t\right]^2$$

where $\alpha = (\gamma-1)/\gamma$.

Problem 4.13
Refer to Figure 6 for the schematic. The flow is two-dimensional, inviscid and steady.

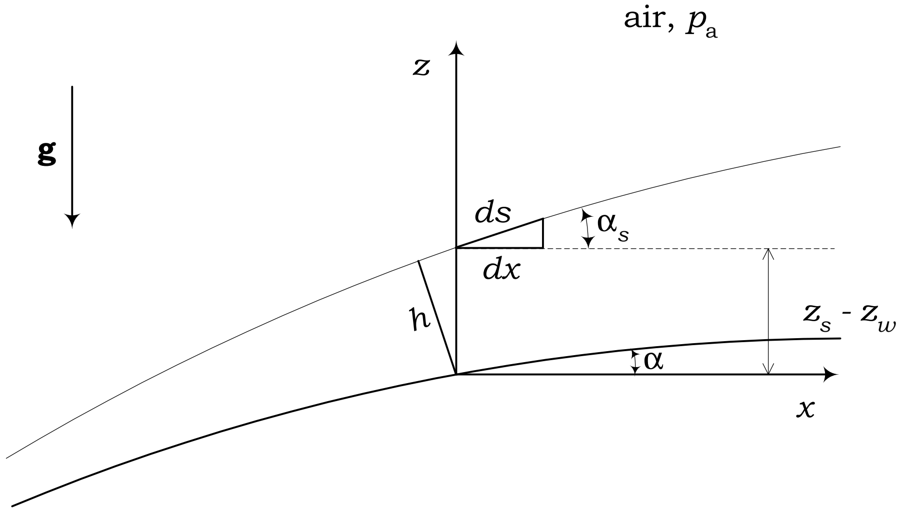

Figure 6: Schematic of Problem 4.13

(a) We consider a stream line along the free surface, Euler's equation in differential form is

$$\frac{1}{\rho}\frac{dp}{ds} + \frac{d}{ds}(gz_s) + \frac{1}{2}\frac{d(u^2)}{ds} = 0$$

where the subscript s denotes the free-surface. Since $p \simeq p_a$ on the surface, then

$$g\frac{dz_s}{ds} + \frac{1}{2}\frac{d(u^2)}{ds} = 0$$

One the surface $ds = dx/\cos\alpha_s$ so that

$$g\frac{dz_s}{dx} + \frac{1}{2}\frac{d(u^2)}{dx} = 0$$

Conservation of mass

$$Q = uh$$
$$\Rightarrow \quad \frac{1}{2}\frac{d(u^2)}{dx} = \frac{1}{2}\frac{d}{dx}\left(\frac{Q^2}{h^2}\right) = -\frac{Q^2}{h^3}\frac{dh}{dx} = -\frac{u^2}{h}\frac{dh}{dx}$$

Noting that $h \simeq z_s - z_w$ then

$$g\frac{dz_s}{dx} \simeq \frac{u^2}{h}\left(\frac{dz_s}{dx} - \frac{dz_w}{dx}\right)$$
$$\Rightarrow \quad \frac{dz_s}{dx} \simeq \frac{\frac{u^2}{gh}}{\frac{u^2}{gh} - 1}\tan\alpha$$

where $tan\alpha = dz_w/dx$.

A Vorticity form of Euler's equation

Euler's equation in differential form is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

We invoke the identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla\, (\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\, \mathbf{u}$$

Noting that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ then

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\frac{1}{\rho}\nabla p - \nabla(gz) - \frac{1}{2}\nabla|\mathbf{u}|^2$$

Next we take the curl of the above equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times \mathbf{u} \times \boldsymbol{\omega} = -\nabla \times \left(\frac{1}{\rho}\nabla p\right)$$

We invoke another identity

$$\nabla \times \mathbf{u} \times \boldsymbol{\omega} = \mathbf{u}(\nabla \cdot \boldsymbol{\omega}) + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$$

Noting that $\nabla \cdot \boldsymbol{\omega} = 0$ and $\nabla \cdot \mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$ from the continuity, then

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt} - \nabla \times \left(\frac{1}{\rho}\nabla p\right)$$

$$\frac{D\boldsymbol{\omega}}{Dt} - \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \nabla \times \left(\frac{1}{\rho}\nabla p\right)$$

We invoke the identity

$$\nabla \times (a\mathbf{A}) = a(\nabla \times \mathbf{A}) + (\nabla a) \times \mathbf{A}$$

so that

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho}(\nabla \times \nabla p) + \nabla\left(\frac{1}{\rho}\right) \times \nabla p = -\frac{1}{\rho^2}\nabla \rho \times \nabla p$$

So that

$$\frac{D\boldsymbol{\omega}}{Dt} - \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \frac{1}{\rho^2}\nabla \rho \times \nabla p$$

Noting that

$$\frac{D\boldsymbol{\omega}}{Dt} - \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt} = \rho\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right)$$

Then

$$\rho\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \frac{1}{\rho^2}\nabla \rho \times \nabla p + \nabla \times \mathbf{F}_b \tag{17}$$

where body force $\mathbf{F}_b$ was included in the equation.

Analysis of equation (17)

- If the body force is conservative ($\nabla \times \mathbf{F}_b = \mathbf{0}$) and the flow is incompressible, then

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \tag{18}$$

- If additionally the flow is steady

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \mathbf{0} \tag{19}$$

- If the body force is conservative ($\nabla \times \mathbf{F}_b = \mathbf{0}$) and the flow is barotropic ($\nabla \rho \times \nabla p = \mathbf{0}$), then

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\right)\mathbf{u} \tag{20}$$