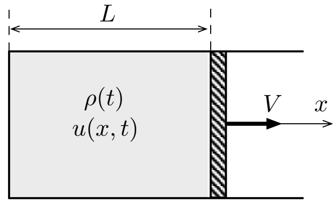
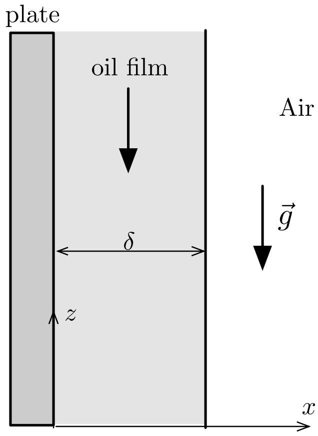
Quiz 2

Problem 1 [34 points]
A gas-filled pneumatic strut in an automobile suspension system can be modeled as a piston in a cylinder with a gas of uniform density, i.e. $\rho = \rho(t)$ only, where t is time. As the piston moves, the gas moves also, and we will assume the velocity in the piston is one-dimensional, i.e. $u = u(x,t), v = w = 0$. Given that the speed of the piston V is constant and that $L = L_0, \rho = \rho_0$ at $t = 0$. Find $u(x,t)$ and $\rho(t)$ in terms of ρ_0, L_0, V, and t.



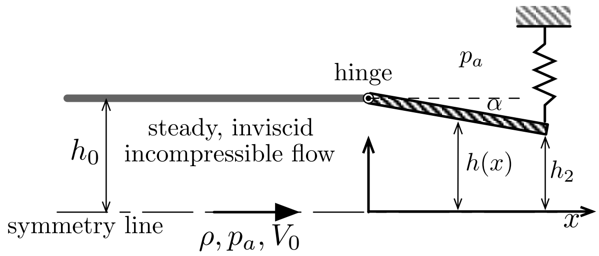
Problem 1.

Problem 2 [33 points]
Oil, of density ρ and viscosity μ, drains steadily down the side of a vertical plate of infinite depth, so that the flow is two-dimensional in the $x-z$ plane as shown. After a development region near the top of the plate, the oil film will become fully developed in the $z-$direction with a constant film thickness δ. Assume that the surrounding air velocity is negligible. In terms of given quantities, find the velocity distribution (u, v, w) the volume flowrate of oil draining down the plate?



Problem 2.

Problem 3 [33 points]
Consider the steady inviscid incompressible flow shown in figure. The flow passes first through a two-dimensional channel of constant height h_0. The flow then passes under a plate of length L. The plate is hinged to the upper wall of the channel while its other end is suspended from a linear elastic string of spring constant k. The spring is under no tension or compression when the plate is horizontal.
(a) Explain why the configuration shown with the plate titled down with angle α is a possible equilibrium state.
(b) Assuming the pressure inside the converging channel is a function of x only, find $p(x)$ in terms of p_a, ρ, V_0, h_0, h_2, L and x.
(c) Find expressions for the reaction forces on the hinge in terms of given quantities?
(d) [BONUS] Find h_2 in terms of p_a, ρ, V_0, h_0, L and k, for $\alpha << 1$.



Problem 3.

Problem 1 Solution
The mass of air inside the sealed cylinder is constant so that $m = \rho_0 L_0 A = \rho(t)L(t)A$. Noting that $V = dL/dt$ then $L(t) = L_0 + Vt$ since V is constant. Then

$$\rho(t) = \rho_0 \frac{L_0}{L_0 + Vt} \tag{1}$$

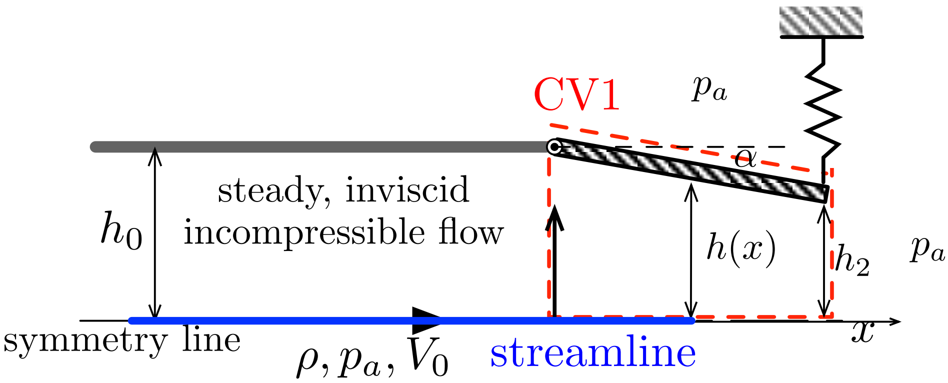
Using the continuity equations and noting that $\rho = \rho(t)$ and $u = u(x,t), v = 0, w = 0$, we get

$$\partial\rho/\partial t + \rho\partial u/\partial x = 0 \Rightarrow \partial u/\partial x = -\frac{1}{\rho}\frac{d\rho}{dt} \Rightarrow u = -\frac{1}{\rho}\frac{d\rho}{dt}x + C(t)$$

Noting that $x = L(t)$, $u = V$, then

$$u = \frac{1}{\rho}\frac{d\rho}{dt}(L - x) + V = x\frac{V}{L_0 + Vt} \tag{2}$$

Problem 3 Solution



Since the flow is inviscid, steady, and incompressible, then we can use Bernoulli's equation along a streamline, which, in the absence of gravitational effects, states that $p + \rho V^2/2$ = constant along a streamline. Taking the streamline from station 0 to station x in the converging part of the channel, we have

$$p_a + \frac{1}{2}\rho V_0^2 = p(x) + \frac{1}{2}\rho V(x)^2 \tag{3}$$

For incompressible steady flow, conservation of mass leads

$$\rho V_0 h_0 = \rho V(x) h(x) \tag{4}$$

So as $h(x)$ decreases, the flows increases it speed to conserve mass, and then from Bernoulli, the pressure must go down. So the pressure in the flow in the converging part of the channel has a pressure less than atmospheric resulting in a net downward force on the channel wall thus deflecting it downwards around the hinge.
(b) Combining Eqs. (5) and (4), and noting that $h(x) = h_0 - x\tan\alpha$, we get

$$p(x) = p_a + \frac{1}{2}\rho V_0^2\left(1 - \frac{h_0^2}{h(x)^2}\right) = p_a + \frac{1}{2}\rho V_0^2\left(1 - \frac{h_0^2}{(h_0 - x\tan\alpha)^2}\right) \tag{5}$$

(c) We take the control volume CV1 and apply conservation of linear momentum in x and y directions, then

$$
\begin{aligned}
x\text{-dir} \qquad & & p_a h_0 - p_2 h_2 - p_a(h_0 - h_2) + R_x = \dot{m}(V_2 - V_0) \\
& \Rightarrow & R_x = \dot{m}V_0\left(\frac{h_0}{h_2} - 1\right) + h_2(p_2 - p_a) \\
& \Rightarrow & R_x = \dot{m}V_0\left(\frac{h_0}{h_2} - 1\right) + h_2\frac{1}{2}\rho V_0^2\left(1 - \frac{h_0^2}{h_2^2}\right) \\
& \Rightarrow & R_x = \dot{m}V_0\left(\frac{h_0}{h_2} - 1\right) + \frac{1}{2}\dot{m}V_0\frac{h_2}{h_0}\left(1 - \frac{h_0^2}{h_2^2}\right) \\
& \Rightarrow & R_x = \dot{m}V_0\left(\frac{1}{2}\frac{h_0}{h_2} + \frac{1}{2}\frac{h_2}{h_0} - 1\right) \\
& \Rightarrow & R_x = \frac{1}{2}\dot{m}V_0\left(\frac{\alpha L}{h_0}\right)^2 \text{ for } \alpha << 1 \qquad (6) \\
y\text{-dir} \qquad & \Rightarrow & \int_0^{L\cos\alpha}(p(x) - p_a)dx - k(h_0 - h_2) + R_y = 0 \\
& \Rightarrow & R_y = \alpha\left(\frac{1}{2}\frac{\rho V_0^2 L^2}{h_0} - kL\right) \text{ for } \alpha << 1 \qquad (7)
\end{aligned}
$$

where R_x and R_y are the reaction force components at the hinge, p_2 and V_2 are the pressure and speed at the channel exit, i.e. at $x = L\cos\alpha$. Note also that $\sin\alpha = (h_0 - h_2)/L$ or $h_2 = h_0 - L\sin\alpha$.