MECH 314 Introduction to Fluid Mechanics September 27, 2009
American University of Beirut, Spring 2009 Handout # 4

Inviscid Flows

1 Euler's Equation for Inviscid Flows

Consider a small particle of fixed identity (control mass) of density ρ, volume $\delta\mathcal{V}$, and velocity $\mathbf{u}$. Applying Newton's second law,

$$\sum \mathbf{F} = \frac{d}{dt}(\rho\,\delta\mathcal{V}\,\mathbf{u})$$

where the forces acting on the particle are

- body forces: weight $\rho\,\delta\mathcal{V}\,\mathbf{g}$, where $\mathbf{g}$ is gravity.
- surface forces: the normal surface force is due to pressure $\int_{C.S.} -p\hat{\mathbf{n}}\,d\mathcal{S} = \int_{C.V.} -\nabla p\,d\mathcal{V} = -\nabla p\,\delta\mathcal{V}$. For an inviscid flow, the viscous tangential force acting on the particle boundary surface is zero.

In the absence of other forces, and employing conservation of mass $d(\rho\,\delta\mathcal{V})/dt = 0$, and dividing by particle volume $\delta\mathcal{V}$, we arrive at Euler's equation

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g} \tag{1}$$

where the term $\rho\frac{d\mathbf{u}}{dt}$, the mass of the particle per unit volume times its acceleration, is the *inertia* term, the term $-\nabla p$ is the pressure force per unit volume, and $\rho\mathbf{g}$ is the particle weight per unit volume.

Note that the inertia term is the sum of a *time-varying* term and a *convective* term

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

If the fluid density is constant, then

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla\hat{p} \tag{2}$$

where where $\hat{p} = p + \rho g z$ and $\mathbf{g} = -g\hat{\mathbf{z}}$.

2 Euler's Equation in Cartesian Coordinates

In Cartesian coordinates, where z points vertically upwards, the components of Euler's equation in the x, y and z directions are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

where u, v and w are the x, y and z components of the velocity $\mathbf{u}$.

3 Euler's Equation in Streamline Coordinates

Euler's equation, when expressed in streamline coordinates, provides useful and physically insightful relationships between the various terms in relation to the streamlines. Let s, n and l be respectively the coordinates along a streamline, normal to a streamline along the radius of curvature, and in the binormal direction, see Fig. 3.

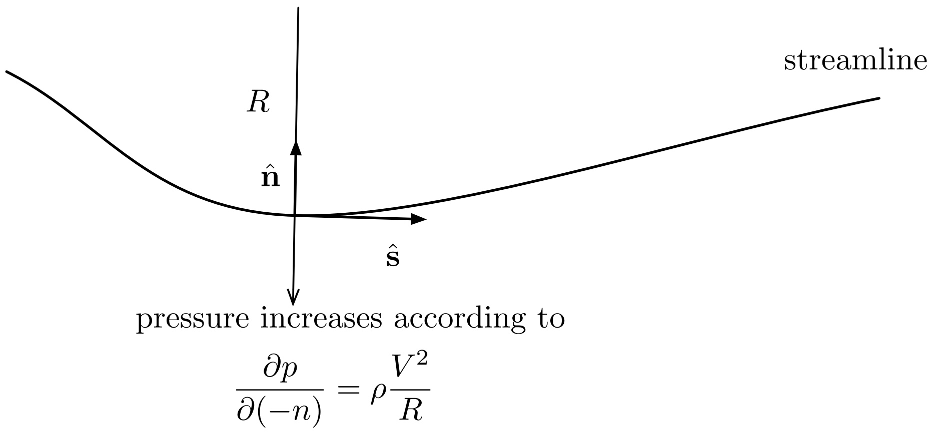

Figure 1: Streamline coordinates

The corresponding unit vectors are $\hat{\mathbf{s}}, \hat{\mathbf{n}}$ and $\hat{\mathbf{l}}$. The components of Euler equation are

$$\begin{aligned} \text{along a streamline} \quad & \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}\right) = -\frac{\partial p}{\partial s} + \rho\mathbf{g}\cdot\hat{\mathbf{s}} \\ \text{normal to a streamline} \quad & \rho\frac{u^2}{R} = -\frac{\partial p}{\partial n} + \rho\mathbf{g}\cdot\hat{\mathbf{n}} \\ \text{binormal to a streamline} \quad & 0 = \frac{\partial p}{\partial l} + \rho\mathbf{g}\cdot\hat{\mathbf{l}} \end{aligned}$$

If the fluid density is constant, then

$$\begin{aligned} \text{along a streamline} \quad & \rho\left(\tfrac{\partial u}{\partial t} + u\tfrac{\partial u}{\partial s}\right) = -\tfrac{\partial \hat{p}}{\partial s} \\ \text{normal to a streamline} \quad & \rho\tfrac{u^2}{R} = -\tfrac{\partial \hat{p}}{\partial n} \\ \text{binormal to a streamline} \quad & 0 = \tfrac{\partial \hat{p}}{\partial l} \end{aligned}$$

Euler's equation normal to a streamline indicates that the pressure in the direction normal to a streamline increases inversely proportional to the radius of curvature of the streamline.

Integrating the equation along the streamline between points 1 and 2 along the streamline,

$$\int_1^2 \frac{\partial u}{\partial t}\, ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = C(t)$$

where $C(t)$ is a function of time. If the fluid density along the streamline is constant, then

$$\int_1^2 \frac{\partial u}{\partial t}\, ds + \frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = C(t) \tag{3}$$

4 Bernoulli's Equation

If the flow is inviscid and steady, and if additionally the density is constant along a streamline, then Eq. (3) reduces to Bernoulli's equation along a streamline

$$p + \frac{1}{2}\rho V^2 + \rho g z = \text{constant}$$

so that for any two points 1 and 2 on the same streamline,

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

5 Criterion for Inviscid Flows

A necessary condition to approximate a flow as inviscid is for the inertia term $\rho\, d\mathbf{u}/dt$ to be much larger, in absolute value, than the viscous force per unit volume, which for a newtonian fluid, is given by $\mu \nabla^2 \mathbf{u}$, so we must have

$$\frac{\text{interia}}{\text{viscous force}} = \frac{|\rho\, d\mathbf{u}/dt|}{|\mu \nabla^2 \mathbf{u}|} >> 1$$

Choosing L and U to be characteristic length and velocity of the flow then

$$\left|\rho \frac{d\mathbf{u}}{dt}\right| \sim \rho \frac{U}{L/U}$$

$$|\mu\, \nabla^2 \mathbf{u}| \sim \mu \frac{U}{L^2}$$

Then a necessary condition for approximating flow as inviscid is

$$\frac{\rho \frac{U}{L/U}}{\mu \frac{U}{L^2}} >> 1 \Rightarrow \mathtt{R}e \equiv \frac{\rho U L}{\mu} >> 1$$

Be aware that a large Reynolds number is only a necessary condition. To apply Bernoulli's equation along a streamline, make sure the following is true

- flow is steady
- density is constant along streamline
- Reynolds number is very large $\mathtt{R}e >> 1$
- stay away from walls, since in actual flows, and no matter how large $\mathtt{R}e$ is, there is always a viscous boundary layer attached to the wall.
- Bernoulli's equation is not applicable in regions of mixing, which are commonly (but not exclusively) encountered when a large $\mathtt{R}e$ flow experiences sudden change in geometry (expansion in particular). A large $\mathtt{R}e$ flow is highly unstable, since viscous forces, being a stabilizing agent, are very small. This means that a disturbance will be amplified causing random oscillations in the local velocity and generating a spectrum of length and time scales ranging from large eddies all the way to eddies so small that their characteristic Reynolds number in small enough for the viscous force to kick in and stabilize the flow. This is Turbulence!

6 Euler's Equation and its Relation with Vorticity

Noting that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \frac{|\mathbf{u}|^2}{2} - \mathbf{u} \times \nabla \times \mathbf{u}$$

where $\nabla \times \mathbf{u}$ is nothing that the vorticity $\boldsymbol{\omega}$, Eq. (1) may be expressed as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \frac{|\mathbf{u}|^2}{2} - \mathbf{u} \times \boldsymbol{\omega} = -\frac{1}{\rho} \nabla p + \mathbf{g}$$

It may then be seen that if the flow is <u>inviscid</u>, <u>steady</u>, <u>incompressible</u>, and <u>irrotational</u>, then Bernoulli's equation applies between ANY two points in the flow

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$$

A flow is irrotational if the vorticity is zero everywhere.