MECH 314 Introduction to Fluid Mechanics | February 19, 2009
American University of Beirut, Spring 2009 | Handout # 4

Integral Relations for a Control Volume

1. (Lecture 1) **Control Volume vs System**
   System: a collection of matter of fixed identity.
   Control Volume: is a volume fixed in the coordinated frame of reference and is bounded a control surface.

2. (Lecture 1) **Fundamentals physical laws for a system in primitive form**
   We are often interested in the rate of change of the amount of a certain physical property contained in a system of volume V. The physical property could be mass, momentum, energy, moment of momentum, entropy etc. The amount of physical property contained in V may be expressed as $B \equiv \int_V b\rho\, dV$ so that the rate of change is $\frac{DB}{Dt}$. The fundamental physical laws governing the rate of change of B is $\frac{DB}{Dt} = \Phi$. The table below presents the physical laws of conservation of mass, momentum and energy expressed in the form $\frac{dB}{dt} = \Phi$.

Conservation Law	b	B	Φ
mass	1	total mass: $\int_V \rho\, dV$	0
linear momentum	$\mathbf{u}$	total momentum: $\int_V \rho \mathbf{u}\, dV$	$\sum \mathbf{F}$
angular momentum	$\mathbf{r} \times \mathbf{u}$	total momentum: $\int_V \rho(\mathbf{r} \times \mathbf{u})\, dV$	$\sum \mathbf{M}_O$
energy	$e + \frac{\lvert\mathbf{u}\rvert^2}{2} + U$	total energy: $\int_V \rho \left(e + \frac{\lvert\mathbf{u}\rvert^2}{2} + U\right)\, dV$	$\sum \delta\dot{q} - \sum \delta\dot{w}$

   where ρ is density, $\mathbf{u}$ is velocity vector, $\mathbf{r}$ is position vector, e is sepcific internal energy, U is potential energy, $\sum \mathbf{F}$ is the sum of all the forces (surface and body) acting on the control volume, $\sum \mathbf{M}_O$ is the sum of all the moments about point O applied to the control volume, $\sum \delta\dot{q}$ is the sum of heat gained by the comtrol volume, and $\sum \delta\dot{w}$ is the sum of work done by the control volume (shaft, pressure, viscous stresses).

3. **The need for a control volume approach and the Reynolds Transport Theorem**
   In fluid mechanics, a system undergoes changes in its shape ans volume. It could experience stretching, twisting, expansion etc. It is therefore difficult to predict V in a Lagrangian sense. It is more convenient to apply the laws of physics for a control volume since it is a fixed volume. In order to apply the above physical laws for a control volume, we need to express the rate of change of the total quantity B in terms of rate of change inside the control volume and fluxes crossing the control surface. In other words:

$$\frac{D}{Dt}\int_V b\rho\, dV \xrightarrow{?} (?)_{\text{C.V.}}$$

From the figure

$$\begin{aligned}\frac{DB}{Dt} &= \frac{(B_I(t+dt)+B_{II}(t+dt))-(B_I(t)+B_{III}(t+dt))}{dt}\\ &= \frac{B_I(t+dt)-B_I(t)}{dt}+\frac{B_{II}(t+dt)}{dt}-\frac{B_{III}(t+dt)}{dt}\\ &= \frac{\partial}{\partial t}B_{\text{CV}}+(\text{flux})_{\text{out}}-(\text{flux})_{\text{in}}\\ &= \frac{\partial}{\partial t}\int_{\text{CV}} b\rho\,dV+\int_{\text{CS}} b\,dw_{\text{out}}-\int_{\text{CS}} b\,dw_{\text{in}}\end{aligned}$$

where w is the flow of quantity b through the control surface. Reynolds Transport Theorem

$$\frac{D}{Dt}\int_{\text{system}} b\rho\,dV = \frac{\partial}{\partial t}\int_{\text{CV}} b\rho\,dV + \int_{\text{CS}} b\,(\rho\,\mathbf{u}\cdot d\mathbf{A})$$

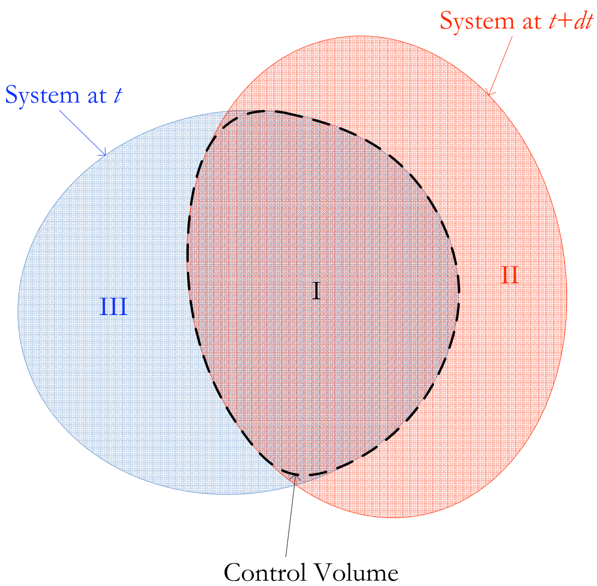

Figure 1: Schematic.

If the control volume (of constant shape) is moving at a constant velocity $\mathbf{u}_{\text{CV}}$, then Reynolds transport theorem assumes the form

$$\frac{D}{Dt}\int_{\text{system}} b\rho\,dV = \frac{\partial}{\partial t}\int_{\text{CV}} b\rho\,dV + \int_{\text{CS}} b\,(\rho\,\mathbf{u}_r\cdot d\mathbf{A})$$

where the relative velocity $\mathbf{u}_r = \mathbf{u} - \mathbf{u}_{\text{CV}}$.

4. (Lecture 2) Convervation of mass (example)

5. (Lecture 2 + Lecture 3) Conservation of linear momentum (example)

6. (Lecture 3) Conservation of angular momentum (example)
7. (Lecture 4) Conservation of energy (example)

Read: White Sections 3.1-3.6