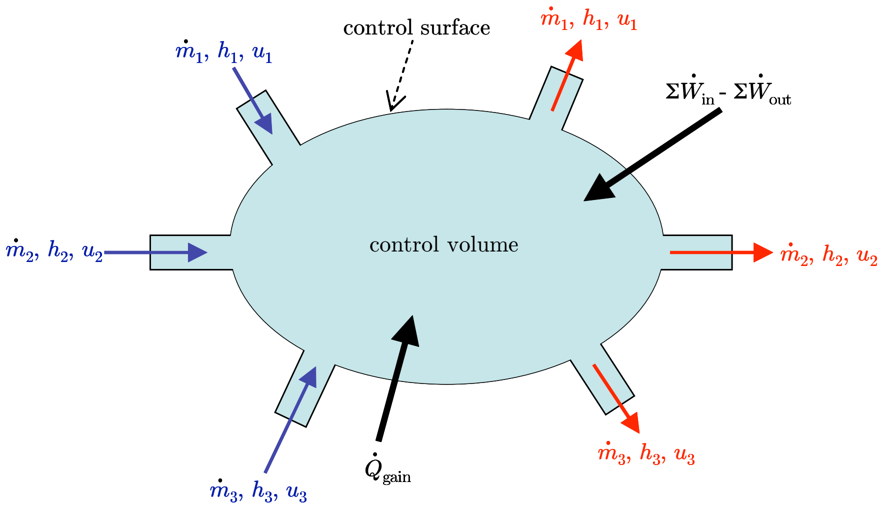
Viscous Flow in Pipe Networks: Energy Balance

1 Integral Form of the Energy Equation for a Control Volume



For a control volume $\mathcal{V}$ bounded by control surface $\mathcal{A}$, the conservation of energy is governed by the following equation

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, e\, d\mathcal{V} + \int_{\mathcal{A}} \rho\, e\,(\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{A} = \dot{Q}^{\leftarrow} - \dot{W}^{\rightarrow}_{\text{bulk}} - \dot{W}^{\rightarrow}_{\text{viscous}} + \sum \dot{W}^{\leftarrow}_{\text{ext}} - \sum \dot{W}^{\rightarrow}_{\text{ext}}$$

where ρ is density, $e \equiv \hat{u} + \frac{|\mathbf{u}^2|}{2} + gz$ is the sum of specific internal energy $\hat{u}$, specific kinetic energy, and specific potential energy respectively. The velocity is $\mathbf{u}$ and the unit outward normal vector at the control surface is $\hat{\mathbf{n}}$. The rate of heat transferred across the boundary <u>into</u> the fluid (Watt) is $\dot{Q}^{\leftarrow}$. The rates of work done <u>by</u> the fluid due to pressure and viscous forces are respectively $\dot{W}^{\rightarrow}_{\text{bulk}}$ and $\dot{W}^{\rightarrow}_{\text{viscous}}$. The rate of work done <u>by</u> the fluid onto external devices (such as shaft work in turbine) is $\sum \dot{W}^{\rightarrow}_{\text{ext}}$ and the work done <u>on</u> the fluid by external devices (such as pumps) is $\sum \dot{W}^{\leftarrow}_{\text{ext}}$.

In order to find an expression for the work done on the fluid by the surrounding due to pressure, we proceed as follows. The force exerted on the fluid per unit volume is $-\nabla p$ so that the total force exerted on the fluid at the control surface is $\int_{\mathcal{V}}(-\nabla p)\, d\mathcal{V} = \int_{\mathcal{A}}(-p)\hat{\mathbf{n}}\, d\mathcal{A}$, where $\mathbf{n}$ is the unit normal vector pointing outward from the fluid toward the surrounding. The force due to pressure on surface element $d\mathcal{A}$ is then $d\mathbf{F} = -p\hat{\mathbf{n}}\,d\mathcal{A}$. If the fluid is moving with a velocity $\mathbf{u}$, then it will cover a distance of $d\mathbf{x} = \mathbf{u}\,dt$ over time dt. Then the work done <u>on</u> the moving fluid by pressure during dt is $dW^{\leftarrow} = \int_{\mathcal{A}} d\mathbf{F}\cdot d\mathbf{x} = dt\,\int_{\mathcal{A}}(-p)\mathbf{u}\cdot\hat{\mathbf{n}}\,d\mathcal{A}$. The rate at which work is done <u>by</u> the fluid is then

$$\dot{W}^{\rightarrow}_{\text{pressure}} = \int_{\mathcal{A}}(p)\mathbf{u}\cdot\hat{\mathbf{n}}\,d\mathcal{A}$$

Notice that for a uniform outflow form a channel, the rate of work done by the fluid is $p\,V\,A$ where V is the speed and A is the cross sectional area of the channel. For a uniform flow into a channel, the rate of work done by the fluid is $-p\,V\,A$.

The viscous work done by the environment on the fluid is found in a similar manner

$$\dot{W}^{\rightarrow}_{\text{viscous}} = \int_{\mathcal{A}}\boldsymbol{\tau}\cdot\mathbf{u}\,d\mathcal{A}$$

where $\boldsymbol{\tau}$ is the viscous stress. Now that at a solid boundary $\dot{W}^{\rightarrow}_{\text{viscous}} = 0$. This is applicable for both inviscid flows ($\boldsymbol{\tau} = \mathbf{0}$) and viscous flows ($\mathbf{u} = 0$ at solid boundary).

Now if we move the work done by bulk flow to the left hand side of the energy equation we get

$$\frac{d}{dt}\int_{\mathcal{V}}\rho\,e\,d\mathcal{V} + \int_{\mathcal{A}}\rho\left(e + \frac{p}{\rho}\right)(\mathbf{u}\cdot\hat{\mathbf{n}})\,d\mathcal{A} = \dot{Q}^{\leftarrow} - \dot{W}^{\rightarrow}_{\text{viscous}} + \sum\dot{W}^{\leftarrow}_{\text{ext}} - \sum\dot{W}^{\rightarrow}_{\text{ext}}$$

Noting that $\hat{u} + p/\rho = \hat{h}$ is the enthalpy, then

$$\frac{d}{dt}\int_{\mathcal{V}}\rho\left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right)\,d\mathcal{V} + \int_{\mathcal{A}}\rho\left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)(\mathbf{u}\cdot\hat{\mathbf{n}})\,d\mathcal{A} =$$
$$\dot{Q}^{\leftarrow} - \dot{W}^{\rightarrow}_{\text{viscous}} + \sum\dot{W}^{\leftarrow}_{\text{ext}} - \sum\dot{W}^{\rightarrow}_{\text{ext}}$$

<u>Special cases</u>

- Steady state:
  For the case of steady flow, the energy equation becomes

$$\int_{\mathcal{A}}\rho\left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)(\mathbf{u}\cdot\hat{\mathbf{n}})\,d\mathcal{A} = \dot{Q}^{\leftarrow} - \dot{W}^{\rightarrow}_{\text{viscous}} + \sum\dot{W}^{\leftarrow}_{\text{ext}} - \sum\dot{W}^{\rightarrow}_{\text{ext}}$$

- Steady uniform flow:
  If the flow is steady and the control surface is is composed of a solid boundary plus inlet and outlet sections such that the properties are (spatially) uniform over each section (see Figure below), then the energy equation becomes

$$\sum\dot{m}^{\rightarrow}\left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)_{\text{out}} - \sum\dot{m}^{\leftarrow}\left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)_{\text{in}} =$$
$$\dot{Q}^{\leftarrow} - \dot{W}^{\rightarrow}_{\text{viscous}} + \sum\dot{W}^{\leftarrow}_{\text{ext}} - \sum\dot{W}^{\rightarrow}_{\text{ext}}$$

2 Steady Single Inlet-Single Outlet Incompressible Flow with negligible heat transfer

The enthalpy is $\hat{h} = \hat{u} + p/\rho$. For a simple compressible matter in the compressed liquid state, the density is constant and the internal energy is approximately a function of temperature, $\hat{u} \simeq \hat{u}(T)$. In the absence of heat transfer across the boundary, the only way the internal energy (temperature) can change is by viscous heat dissipation of the work done to overcome friction. If we neglect this (usually very small) change in internal energy, then for a steady single inlet-single single-outlet incompressible flow with no heat transfer, the energy equation for the control volume shown in Fig. 1 is

$$\int_{\mathcal{A}_2} \left(p + \rho\frac{|u^2|}{2} + \rho g z\right) u \, d\mathcal{A} - \int_{\mathcal{A}_1} \left(p + \rho\frac{|u^2|}{2} + \rho g z\right) u \, d\mathcal{A} = -\dot{W}^{\rightarrow}_{\text{viscous}} + \sum \dot{W}^{\leftarrow}_{\text{ext}} - \sum \dot{W}^{\rightarrow}_{\text{ext}}$$

We assume that the pressure is uniform at cross sections 1 and 2. Expressing $u =$

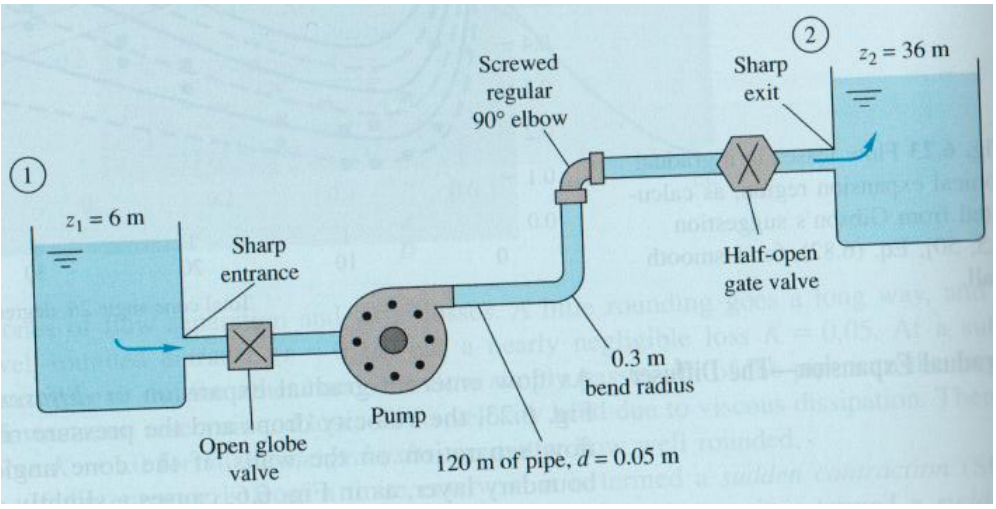


Figure 1: Sample Control Volume

$V + u'$, $z = Z + z'$, where V and Z are respectively the average velocity and cross sectional elevation: $V = \frac{1}{\mathcal{A}}\int_{\mathcal{A}} u d\mathcal{A}$ and $Z = \frac{1}{\mathcal{A}}\int_{\mathcal{A}} z d\mathcal{A}$, then after dividing by $\dot{m}g$, where

$$\dot{m} = \int_{\mathcal{A}_2} \rho u \, d\mathcal{A} = \int_{\mathcal{A}_1} \rho u \, d\mathcal{A}$$

we get

$$\frac{p_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + \beta_1 Z_1 = \frac{p_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + \beta_2 Z_2 + \frac{\dot{W}^{\rightarrow}_{\text{viscous}}}{\dot{m}g} - \sum \frac{\dot{W}^{\leftarrow}_{\text{ext}}}{\dot{m}g} + \sum \frac{\dot{W}^{\rightarrow}_{\text{ext}}}{\dot{m}g}$$

where

$$\alpha = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} \left(1 + 3\frac{u'^2}{V^2} + \frac{u'^3}{V^3}\right) d\mathcal{A}$$
$$\beta = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} \left(1 + \frac{u'z'}{VZ}\right) d\mathcal{A}$$

The external work into the control volume (other than *pdv* work) could be from a pump so that the pump head is

$$h_p = \sum \frac{\dot{W}^{\leftarrow}_{\text{ext}}}{\dot{m}g}$$

The external work out of the control volume (other than *pdv* work) could be shaft work in a turbine so that the turbine head is

$$h_t = \sum \frac{\dot{W}^{\rightarrow}_{\text{ext}}}{\dot{m}g}$$

The viscous work accounts for losses due to skin friction along the channels h_f and minor losses in bends, valves, elbows, etc, so that losses head is

$$h_f + \sum h_m = \frac{\dot{W}^{\rightarrow}_{\text{viscous}}}{\dot{m}g}$$

Assuming $\beta_1 = \beta_2 = 1$, then

$$\frac{p_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + Z_2 + h_f + \sum h_m - h_p + h_t \tag{1}$$

Note that for uniform flow $\alpha = 1$, for fully developed parabolic (laminar) flow $\alpha =$, and for fully developed turbulent flow $\alpha =$.

2.1 Head Loss in a pipe (h_f) - The Friction Factor

For fully developed flow in a circular pipe of diameter d and length L (no minor losses, no turbine work, no pump work, $V_1 = V_2$, $\alpha_1 = \alpha_2$) shown in Fig. 2, we get from Eq. (1)

$$h_f = \Delta Z + \frac{\Delta p}{\rho g} \tag{2}$$

where $\Delta p = p_1 - p_2$ and $\Delta Z = Z_1 - Z_2$. Applying the conservation of momentum in the flow direction for a control volume of length L and diameter d, the net pressure force, gravity, and friction force are related, at steady conditions, by

$$\Delta p(\pi R^2) + \rho g(\pi R^2)L \sin\phi - \tau_w(2\pi RL) = \dot{m}(V_2 - V_1) = 0$$

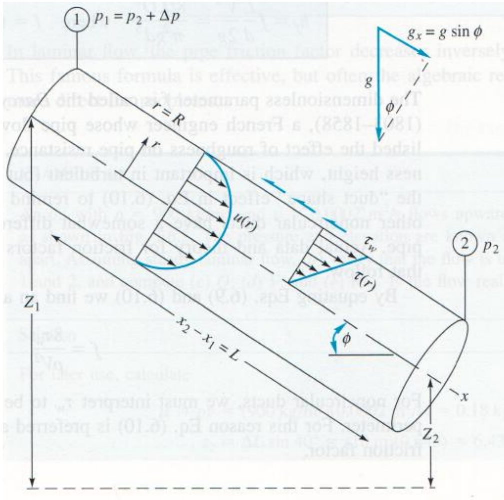


Figure 2: Control volume for pipe section

Then

$$h_f = \frac{\Delta p}{\rho g} + \Delta Z = \frac{4\tau_w}{\rho g}\frac{L}{d} \equiv f\frac{L}{d}\frac{V^2}{2g} \tag{3}$$

where f is the friction factor, which is presented next for fully developed laminar and turbulent pipe incompressible pipe flows.

<u>Laminar Flow</u>:
Recall that for a Poiseuille (laminar fully developed steady incompressible flow in a circular pipe),

$$u = 2V\left(1 - \frac{r^2}{R^2}\right), \text{ where } V = \frac{1}{2}\left(\frac{\Delta p + \rho g \Delta Z}{L}\right)\frac{R^2}{4\mu} \tag{4}$$

Noting that $\tau_w = |\mu du/dr|_{r=R}$, we get for laminar flow

$$f_{\text{laminar}} = \frac{64}{Re_d} \tag{5}$$

<u>Turbulent Flow</u>:
For turbulent flow, we expect from dimensional analysis

$$h_f = h_f(\rho, \mu, V, L, d, \epsilon) \Rightarrow f = f\left(Re_d, \frac{\epsilon}{d}\right)$$

where ϵ is the surface roughness in m, listed in Table 3 for different materials. From experimental correlation,

$$\frac{1}{f_{\text{turbulent}}^{1/2}} = -2.0\log\left(\frac{\epsilon/d}{3.7} + \frac{2.51}{Re_d f_{\text{turbulent}}^{1/2}}\right) \qquad \text{Colebrook, Moody's Chart} \tag{6}$$

$$\frac{1}{f_{\text{turbulent}}^{1/2}} \simeq -1.8 \log \left(\left(\frac{\epsilon/d}{3.7} \right)^{1.11} + \frac{6.9}{Re_d} \right) \qquad \text{Haaland} \tag{7}$$

Note that Moody's chart, shown in Fig. 4 is in Figure 6.13 in the book.

2.2 Minor Losses

In addition to the friction losses in fully developed flow along straight constant area pipe sections, there are losses at (i) pipe entrance or exit, (ii) sudden expansion or contractions, and in (iii) bends, elbows, tees and other fittings, (iv) valves, open or partially closed, and (5) gradual expansions or contractions. These minor losses are expressed as

$$\sum h_m = \sum \frac{(\Delta p)_i}{\rho g} = \sum \left(\frac{V^2}{2g} K \right)_i \tag{8}$$

where $(\Delta p)_i$ is the pressure loss across component i, and K_i is the loss coefficient of component i, defined as

$$K_i = \left(\frac{h_m}{V^2/2g} \right)_i \tag{9}$$

Note that if the pipe diameter does not change in the entire system,

$$\sum h_m = \sum \frac{(\Delta p)_i}{\rho g} = \frac{V^2}{2g} \sum K \tag{10}$$

The loss coefficients for different components are provided in tables, charts, or figures. See in the Book.

3 The Bottom Line

So for a steady single inlet-single single-outlet incompressible flow with no heat transfer, the energy equation for the control volume shown is

$$\frac{p_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + Z_2 + \sum_i \left(f_i \frac{L_i}{d_i} \frac{V_i^2}{2g} \right) + \sum_j \left(K_j \frac{V_j^2}{2g} \right) - h_p + h_t$$

where summation i is over all pipe sections of different diameters, and summation j is over all components that contribute to minor losses.

Material	Condition	ε ft	ε mm	Uncertainty, %
Steel	Sheet metal, new	0.00016	0.05	±60
	Stainless, new	0.000007	0.002	±50
	Commercial, new	0.00015	0.046	±30
	Riveted	0.01	3.0	±70
	Rusted	0.007	2.0	±50
Iron	Cast, new	0.00085	0.26	±50
	Wrought, new	0.00015	0.046	±20
	Galvanized, new	0.0005	0.15	±40
	Asphalted cast	0.0004	0.12	±50
Brass	Drawn, new	0.000007	0.002	±50
Plastic	Drawn tubing	0.000005	0.0015	±60
Glass	—	Smooth	Smooth	
Concrete	Smoothed	0.00013	0.04	±60
	Rough	0.007	2.0	±50
Rubber	Smoothed	0.000033	0.01	±60
Wood	Stave	0.0016	0.5	±40

Figure 3: Surface roughness for materials.

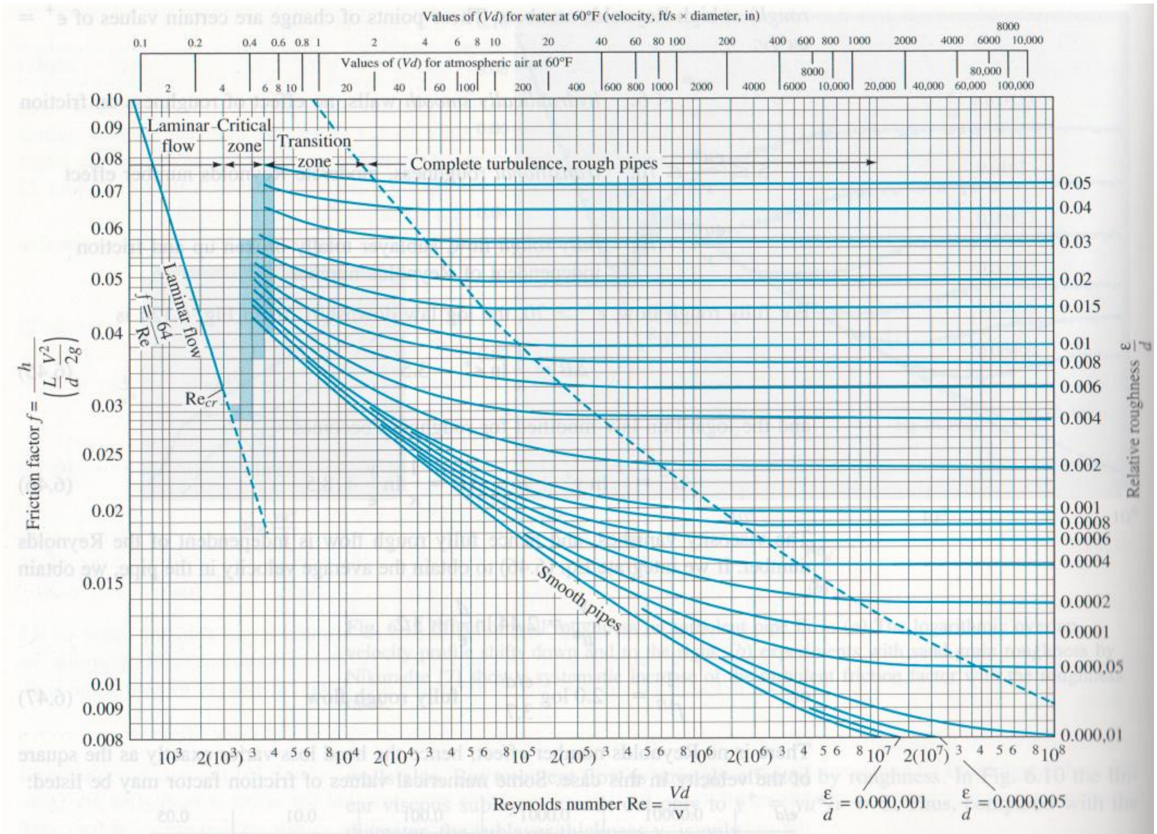


Fig. 6.13 The Moody chart for pipe friction with smooth and rough walls. This chart is identical to Eq. (6.48) for turbulent flow. *(From Ref. 8, by permission of the ASME.)*

Figure 4: Moody's Chart