MECH 314 Introduction to Fluid Mechanics | April 26, 2009
American University of Beirut, Spring 2009 | Handout # 5

Incompressible Flow Criterion

A flow is incompressible if

$$\left|\frac{1}{\rho}\frac{d\rho}{dt}\right| << \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| + \left|\frac{\partial w}{\partial z}\right|$$

or

$$\left|\frac{d\rho}{\rho}\right| << \left(\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| + \left|\frac{\partial w}{\partial z}\right|\right) dt$$

The term on the right hand side scales as one, since $|\partial u/\partial x| \sim U/L$ and $\delta t \sim L/U$, where L and U are respectively characteristic length and velocity. So a flow is approximated as incompressible if density change $\delta\rho$ over an infinitesimal process t to $t + \delta t$ is very small compared to the density ρ. Under the condition

$$\left|\frac{\delta\rho}{\rho}\right| << 1 \Rightarrow \nabla \cdot \mathbf{u} \simeq 0$$

For a simple single-phase fluid, the thermodynamic state is defined by two independent variables. Choosing pressure and specific entropy as these independent variables, then

$$\begin{aligned} &\rho = \rho(p, s) \\ \Rightarrow \quad &d\rho = \left(\frac{\partial\rho}{\partial p}\right)_s dp + \left(\frac{\partial\rho}{\partial s}\right)_p ds \end{aligned}$$

The term $\left(\frac{\partial\rho}{\partial p}\right)_s$, denoting the change in density cause by change in pressure during an infinitesimal isentropic process, is the inverse of the speed of sound (a) in the fluid squared

$$\left(\frac{\partial\rho}{\partial p}\right)_s \equiv \frac{1}{a^2} \Rightarrow \delta\rho = \frac{\delta p}{a^2}$$

So for an isentropic infinitesimal process, the following condition for incompressibility must hold

$$\left|\frac{\delta p}{\rho}\right| << a^2 \tag{1}$$

For the flow to be isentropic, it must necessarily be inviscid (why?), so that

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

so that

$$\frac{U}{(L/U)} \sim \frac{\delta p/L}{\rho} \Rightarrow \left|\frac{\delta p}{\rho}\right| \sim U^2$$

Combining with eq. (1), the condition for incompressibility for an isentropic infinitesimal process reduces to

$$\frac{U^2}{a^2} << 1 \qquad \text{incompressible flow condition}$$

Recognizing the ratio of flow speed to speed of sound in the fluid as Mach number $\texttt{Ma}$, then

$$\texttt{Ma} < 0.2 \qquad \text{incompressible flow condition}$$

Challenge

Show that the speed of sound in an ideal gas is $a = \sqrt{\gamma RT}$, where $\gamma = c_p/c_v$.