A Note on the Integral form of the Conservation of Energy for a Control Volume

For a control volume $\mathcal{V}$ bounded by control surface $\mathcal{S}$, the conservation of energy is governed by the following equation

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, e\, d\mathcal{V} + \int_{\mathcal{S}} \rho\, e\, (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} = \dot{Q}_{\text{gained}} + \dot{W}_{\text{pressure}} + \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

where ρ is density, $e \equiv \hat{u} + \frac{|\mathbf{u}^2|}{2} + gz$ is the sum of specific internal energy $\hat{u} = c_v T$, specific kinetic energy, and specific potential energy respectively. The velocity is $\mathbf{u}$ and the unit outward normal vector at the control surface is $\hat{\mathbf{n}}$. The rate of heat gain by the fluid (Watt) is $\dot{Q}_{\text{gained}}$. The rates of work done by pressure and viscous forces on the fluid are respectively $\dot{W}_{\text{pressure}}$ and $\dot{W}_{\text{shear}}$. The work extracted from the fluid, shaft work to turbine for example, is $\sum \dot{W}^{\text{ext}}_{\text{by fluid}}$ and the work done on the fluid by external devices such as pumps is $\sum \dot{W}^{\text{ext}}_{\text{on fluid}}$.

In order to find an expression for the work done on the fluid by the surrounding due to pressure, we proceed as follows. The force exerted on the fluid per unit volume is $-\nabla p$ so that the total force exerted on the fluid at the control surface is $\int_{\mathcal{V}}(-\nabla p)\, d\mathcal{V} = \int_{\mathcal{S}}(-p)\hat{\mathbf{n}}\, d\mathcal{S}$, where $\mathbf{n}$ is the unit normal vector pointing outward from the fluid toward the surrounding. The force due to pressure on surface element $d\mathcal{S}$ is then $d\mathbf{F} = -p\hat{\mathbf{n}}\, d\mathcal{S}$. If the fluid is moving with a velocity $\mathbf{u}$, then it will cover a distance of $d\mathbf{x} = \mathbf{u}\, dt$ over time dt. Then the work done by the moving fluid during dt is $dW = \int_{\mathcal{S}} d\mathbf{F}\cdot d\mathbf{x} = dt\, \int_{\mathcal{S}}(-p)\mathbf{u}\cdot\hat{\mathbf{n}}\, d\mathcal{S}$. The rate at which work is done on the fluid is then

$$\dot{W}_{\text{pressure}} = \int_{\mathcal{S}}(-p)\mathbf{u}\cdot\hat{\mathbf{n}}\, d\mathcal{S}$$

Notice that for a uniform outflow form a channel, the rate of work done on the fluid is $-p\,V\,A$ where V is the speed and A is the cross sectional area of the channel. For a uniform flow into a channel, the rate of work done on the fluid is $p\,V\,A$.

The viscous work done by the environment on the fluid is found in a similar manner

$$\dot{W}_{\text{viscous}} = \int_{\mathcal{S}} \boldsymbol{\tau}\cdot\mathbf{u}\, d\mathcal{S}$$

where $\boldsymbol{\tau}$ is the viscous stress. Now that at a solid boundary $\dot{W}_{\text{viscous}} = 0$. This is applicable for both inviscid flows ($\boldsymbol{\tau} = \mathbf{0}$) and viscous flows ($\mathbf{u} = 0$ at solid boundary).

Now if we move the work done on the fluid due to pressure to the left hand side of the energy equation we get

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\, e\, d\mathcal{V} + \int_{\mathcal{S}} \rho \left(e + \frac{p}{\rho}\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} = \dot{Q}_{\text{gained}} + \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

Noting that $\hat{u} + p/\rho = \hat{h}$ is the enthalpy, then

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \left(\hat{u} + \frac{|\mathbf{u}^2|}{2} + gz\right) d\mathcal{V} + \int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} =$$
$$\dot{Q}_{\text{gained}} + \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

Special cases

- Steady state:
  For the case of steady flow, the energy equation becomes

$$\int_{\mathcal{S}} \rho \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right) (\mathbf{u}\cdot\hat{\mathbf{n}})\, d\mathcal{S} = \dot{Q}_{\text{gained}} + \dot{W}_{\text{viscous}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$

- Steady uniform flow:
  If the flow is steady and the control surface is is composed of a solid boundary plus inlet and outlet sections such that the properties are (spatially) uniform over each section (see Figure below), then the energy equation becomes

$$\sum \dot{m}_{\text{out}} \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)_{\text{out}} - \sum \dot{m}_{\text{in}} \left(\hat{h} + \frac{|\mathbf{u}^2|}{2} + gz\right)_{\text{in}} =$$
$$\dot{Q}_{\text{gained}} + \sum \dot{W}^{\text{ext}}_{\text{on fluid}} - \sum \dot{W}^{\text{ext}}_{\text{by fluid}}$$