
The Continuum Hypothesis

1 The continuum hypothesis

Fluid mechanics deals with the behavior of fluids on a macroscopic scale large compared with the characteristic molecular dimensions. For a gas the characteristic molecular dimension is the mean free path whereas for a liquid the characteristic molecular dimension is the average inter-molecular spacing. If we consider the distribution of mass on the molecular scale, we observe that the density of a small volume varies drastically as the volume increases to contain an increasing number of molecules (Figure 1). Such a behavior is expected since the mass of an atom is concentrated in the nucleus, with the latter occupying a very small fraction of the total volume of the molecule. Macroscopic properties such as temperature, pressure, and velocity are defined by averaging over a large number of molecules contained in a small volume. In the continuum hypothesis it is assumed that the macroscopic behavior expressed in terms of these macroscopic properties is such that the fluid behaves as if these properties are uniformly distributed within the small volume.

When we experimentally measure a macroscopic property at point in the fluid, the measurement in general corresponds to the average (spatial and temporal) behavior of an aggregate of molecules contained in a small volume comprising the vicinity of that point. For example, by associating temperature T with point \mathbf{x} , we are making the approximation that T is the average temperature of a large number of molecules contained in a small volume δV . Any further reduction of this volume, let's say from δV to $\delta V'$, should yield the same temperature measurement as long as $\delta V'$ is still much larger than the characteristic molecular dimension cubed. Therefore the continuum hypothesis is based on the possibility of the associating with the small volume δV macroscopic quantities that are considered to be uniform within that volume such that

$$\lambda \ll (\delta V)^{1/3} \ll L \quad (1)$$

where L is the characteristic macroscopic length scale of the flow (such as the pipe diameter or boundary layer thickness).

As an example consider the flow of air in a pipe of diameter of 1 cm. If we choose $(\delta V)^{1/3} = 1$ mm, then an instrument with a sensitive volume of $\delta V = 10^{-9}$ cm³ would obviously provide a measurement of a local macroscopic property since $L/(\delta V)^{1/3} = 1000$. The small volume δV , however, contains 3×10^{10} molecules of air, which is large enough to guarantee that the average measurement is independent of the number of molecules. There are cases in which the continuum hypothesis fails. One example is air flow at very

high altitudes. At such altitudes the density is so small that the mean free path is of the order of magnitude of the thickness of the boundary layer developing on a solid boundary. Another case is the shock wave where density gradients are so large that it is impossible to find a small volume that contains a sufficiently large number of molecules and within which the density may be assumed to be uniform.

The continuum model described above enables field representation of the fluid, i.e. macroscopic properties are continuous functions of time and space throughout the fluid. The continuum hypothesis thus enables establishment of equations governing the fluid flow that is independent in its general form from the particular molecular structure of a fluid. Therefore the same set of equations governs the conservation of mass and momentum for all fluids; liquids and gases. There is still the challenge of determining the physical material properties (such as conductivity, density, viscosity) of the continuum fluid that would yield the same behavior as the real fluid. Such continuum material properties are usually obtained either from simplified molecular analysis such as the kinetic theory of gases or empirically.

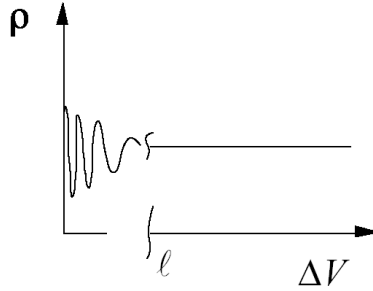


Figure 1: Schematic variation of average density as a function of length scale.

Another assumption is the local thermodynamic equilibrium (LTE). The Navier-Stokes equations are valid only when the flow is in a state of local thermodynamic equilibrium. When a fluid in LTE experiences some kind of perturbation such as a pressure gradient, the fluid is not in equilibrium anymore and it will take some time before it reaches its new LTE state. We refer to this time scale as τ_{th} . Typically τ_{th} is of the order of few tens or hundreds of collision time for translational modes and considerably more for other modes. If the flow time scale τ_{fl} is sufficiently larger than τ_{th} , then the assumption of LTE is valid. In other words the flow macroscopic process must be slow when compared to the time required to adjust to the new LTE state.

In summary the continuum hypothesis is based on the following assumptions:

$$\lambda \ll \delta x \ll L \quad (2)$$

$$\tau_{th} \ll \tau_{fl} \quad (3)$$

A measure of the degree of applicability of the continuum hypothesis is the Knudsen number, defined as the ratio of the mean free path to the the flow characteristic length scale:

$$\text{Kn} = \frac{\lambda}{L} \quad (4)$$