

---

## Conservation Laws for Mass, Momentum, Energy and the Non-Conservation of Entropy

---

Note that bold letters and symbols refer to vectors.

### 1 Subject of Study

Any scientific analysis is incomplete without precise specification of the subject of study. When applying the laws of thermodynamics, specification of the subject of study is usually the first thing to do. When the subject of study is matter of fixed identity, i.e. it always consists of the same molecules, then we are referring to a *control mass*. Examples include a block of copper, the gas inside a sealed container, etc. Tracking the same molecules is not always an easy task and is at times impossible. Consider, for example combustion of fuel and air in the combustion chamber of a gas turbine. Upon combustion, the identity of molecules change. Not only that, molecules which are at time  $t$  inside the combustion chamber moved to somewhere else, say the turbine, at  $t + \Delta t$ . So if we are interested in what happens inside the combustor, we find ourselves chasing molecules far from the combustor at later times. So it makes sense in this case, to take the combustor itself (or the volume enclosed by the combustor walls) as the subject of study, with (fuel + air) molecules flowing in across the boundary and products molecules flowing out across the combustor boundary. This is an example of a *control volume*; a predefined volume that usually contains a region of interest or a device of interest. The boundary of the control volume across which material flows is called the *control surface*.

### 2 Conservation Laws in Thermal and Fluid Sciences

1. conservation of mass (continuity)
2. conservation of momentum (Newton's second law)
3. conservation of energy (first law of thermodynamics)
4. non-conservation of exergy (second law of thermodynamics)

The field of thermal and fluid sciences consists usually of the following interconnected areas: *thermodynamics* (MECH 310, laws 1, 3 and 4), *fluid mechanics* (MECH 410, laws 1 and 2), and *heat transfer* (MECH, laws 1 and 3).

## 3 Conservation of Mass

### 3.1 Conservation of Mass for a Control Mass

Since a control mass (C.M.) consists of matter of fixed identity, then by definition the mass of a control mass is conserved, i.e.

$$\frac{dm_{\text{C.M.}}}{dt} = 0 \Rightarrow \frac{d}{dt} \int_{\mathcal{V}} \rho d\mathcal{V} = 0 \quad (1)$$

where  $\rho$  is the density ( $\text{kg/m}^3$ ) is generally a function of time and space. The volume of the control mass is  $\mathcal{V}$  and in cartesian coordinates  $d\mathcal{V} = dx dy dz$ . In equation (1), one should not mistake  $d()/dt$  for  $\partial()/\partial t$ ; the latter is the derivative in time holding everything else fixed, i.e. at a constant position. The *total* derivative  $d\Pi/dt$  is the change of property  $\Pi$  in time as the particle is moving.

### 3.2 Conservation of Mass for a Control Volume

#### 3.2.1 Integral Form

Consider a control volume  $\mathcal{V}$  bounded by control surface  $\mathcal{S}$ . The integral form of the conservation of mass states that *the rate of accumulation of mass inside the control volume is balanced by the surface integral of mass flux crossing the control surface*

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} = 0 \quad (2)$$

and where  $t$  is time, density is inverse of specific volume  $\rho \equiv \frac{1}{v}$ ,  $\mathbf{u}$  is the velocity vector. The term  $\hat{\mathbf{n}}$  is the unit normal vector on the control surface pointing away from the control volume, as shown in Figure 1. Note that the quantity  $\rho \mathbf{u} \cdot \hat{\mathbf{n}}$  is the mass flux (mass per unit area per unit time) leaving the control volume across the control surface. We may

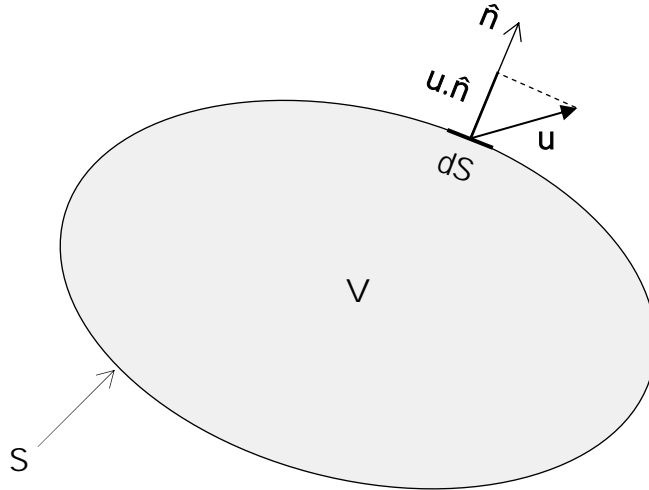


Figure 1: Schematic for conservation of mass for a control volume.

generalize the concept of flux to calculate properties carried by bulk flow crossing the control surface of a control volume. For example, the momentum flux is equal to the

mass flux times the velocity  $\mathbf{u}$ , the total energy flux is equal to the mass flux times the specific total energy  $e$ , the entropy flux is equal to the mass flux times the specific entropy  $s$ .

### 3.2.2 Differential Form

Using Gauss's theorem, we may convert the surface integral to a volume integral as follows:

Consider volume  $\mathcal{V}$  (in 3D space) which is compact and has a piecewise smooth boundary  $\partial\mathcal{V}$ , then for a continuous and differentiable vector field  $\mathbf{F}(\mathbf{x})$ , the following identity holds

$$\int_{\mathcal{V}} \nabla \cdot (\mathbf{F}) d\mathcal{V} = \int_{\partial\mathcal{V}} \mathbf{F} \cdot \hat{\mathbf{n}} d\mathcal{S}$$

Setting  $\mathbf{F} = \rho\mathbf{u}$ , Eq. (5) can then be written as

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho d\mathcal{V} + \int_{\mathcal{V}} \nabla \cdot (\rho\mathbf{u}) d\mathcal{V} = 0$$

Since in our choice of coordinates, space and time are independent, then

$$\int_{\mathcal{V}} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) \right) d\mathcal{V} = 0$$

The above equation holds for any choice of control volume! Then the integrand itself must be zero

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0 \quad (3)$$

Eq. (3) is the differential form of the conservation of mass law; it is also known as the continuity equation.

## 4 Conservation of momentum: Newton's Second Law

### 4.1 Conservation of momentum: Control Mass

Newton's second law states that for a rigid body, the summation of forces acting on the body is equal to its mass times the acceleration. If the body is not rigid, but consists of matter of fixed identity, then conservation of momentum for a control mass states that *the sum of forces acting on a control mass is equal to the rate of change its momentum*. If we decompose the control mass into small volumes, then the momentum of fluid contained in differential volume  $d\mathcal{V}$ , centered at  $\mathbf{x}$ , is equal to the mass contained in  $d\mathcal{V}$  times the velocity<sup>1</sup>  $\mathbf{u}(\mathbf{x})$ , i.e.  $(dm)\mathbf{u} = (\rho d\mathcal{V})\mathbf{u}$ . The total momentum contained in the control mass is then  $\int_{\mathcal{V}} \rho\mathbf{u} d\mathcal{V}$ . The conservation of momentum for a control mass, in integral form, is then expressed as

$$\sum \mathbf{F}_{C.M.} = \frac{d}{dt} \int_{\mathcal{V}} \rho\mathbf{u} d\mathcal{V} \quad (4)$$

---

<sup>1</sup>Since  $d\mathcal{V}$  is very small, we assume that density and velocity are uniform over  $d\mathcal{V}$  and are equal to the density and velocity at its center  $\mathbf{x}$

## 4.2 Conservation of momentum: Control Volume

For a control volume, we must also include the rate of momentum flow across the control surface carried by bulk flow. This is nothing but the moment flux integrated over the control surface. So, for a control volume

$$\sum \mathbf{F}_{c.v.} = \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \mathbf{u} d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u} (\mathbf{u} \cdot \hat{\mathbf{n}}) d\mathcal{S} \quad (5)$$

## 4.3 Forces

One classification of forces is according to whether they act throughout the volume, or on a surface, or along a line interface. This classification arises from the molecular nature of these forces.

### 4.3.1 Body Forces

A body force is a force that is proportional to the volume. More specifically, the body force acting on a differential volume  $d\mathcal{V}$  is proportional to  $d\mathcal{V}$ , i.e.  $d\mathbf{F}_b \propto d\mathcal{V}$ . Example is gravity, in this case,  $d\mathbf{F}_b = \rho \mathbf{g} d\mathcal{V}$ , where  $\mathbf{g}$  is gravity so that  $\mathbf{F}_b = m\mathbf{g}$ . Other examples of body forces are centrifugal and centripetal forces, and coriolis force induced by motion in a rotating frame of reference.

### 4.3.2 Surface Forces

Surface forces are forces that arise from momentum exchange between atoms (molecules) and neighboring atoms (molecules). The atoms (molecules) exchanging momenta by collision type interactions could belong to the same matter or to different matter. For example, the stress state inside a fluid arise from exchange of momentum among atoms (molecules) of the same fluid. On the other hand, in the example of fluid flow over a boundary such airflow over airplane wing, momenta exchanges between atoms (molecules) of the fluid and atoms (molecules) of the solid boundary give rise to normal (pressure) and tangential (shear) forces on the boundary.

The *pressure* at a given position  $\mathbf{x}$  in a fluid is the statistical average of the normal component of the force imparted by molecular momenta exchange across a fictitious plane surface passing through  $\mathbf{x}$  per unit area. The interesting thing about pressure is that the magnitude of this normal force component is independent of the orientation of the plane surface. It is, however, a function of position, i.e.  $p = p(\mathbf{x})$ , i.e. the pressure generally changes with position. For example, as air moves over a wing, the pressure at a point slightly above the wing is smaller than atmospheric, whereas the pressure at a point slightly below the wing is a higher than atmospheric. One may observe that the force due to pressure acting on differential control surface  $d\mathcal{S}$  is  $-p\hat{\mathbf{n}} d\mathcal{S}$ . Note that the negative sign is due to the fact the pressure always acts to compress your control volume where  $\hat{\mathbf{n}}$  is the unit normal vector pointing away from the control volume. The total pressure force acting on a control volume or a control mass is then

$$\mathbf{F}_p = \int_{\mathcal{S}} -p\hat{\mathbf{n}} d\mathcal{S} \quad (6)$$

It may also be shown that, equivalently,

$$\mathbf{F}_p = \int_{\mathcal{V}} -\nabla p d\mathcal{V} \quad (7)$$

Please note that while pressure is a surface force per unit area,  $-\nabla p$  is body force per unit volume.

The components of the momenta exchange among the molecules that are tangent to the surface of interest give rise to viscous or shear stresses in a fluid. We call this force  $\mathbf{F}_{shear}$ . We do not need to know how it looks like quantitatively at this point. One may think of shear or the viscous force as rubbing among adjacent layers of fluid. The outcome of this rubbing is propagation of retardation originating at a fixed solid wall at which the velocity must be zero. The stationary fluid molecules at the wall end up retarding those next to the wall then due to momentum exchange. And those next to the wall end up retarding those next to them on the fluid interior and so on. Obviously, this results in reduction in kinetic energy with an equivalent dissipation into heat. The process is obviously irreversible. Why?

### 4.3.3 Line Forces

Surface tension. Not discussed in this course.

## 5 Conservation of energy: First Law of Thermodynamics

### 5.1 First Law for a Control Mass

For a control mass, no bulk flow is allowed to cross the boundary so that the only means of transferring energy to (from) control mass across the boundary is via heat and/or work, so that

$$\frac{dE_{C.M.}}{dt} = \dot{Q}_{\mathcal{V}}^{\leftarrow} + \dot{W}_{\mathcal{V}}^{\leftarrow} \quad (8)$$

Defining  $e$  (kJ/kg) as the specific total energy (total energy per unit mass), then

$$E = \int_{\mathcal{V}} \rho e d\mathcal{V} \quad (9)$$

so that

$$\frac{d}{dt} \int_{\mathcal{V}} \rho e d\mathcal{V} = \dot{Q}_{\mathcal{V}}^{\leftarrow} + \dot{W}_{\mathcal{V}}^{\leftarrow} \quad (10)$$

where  $\mathcal{V}$  is the volume of the control mass.

Note that  $E$  is the total energy which is the sum of the kinetic energy, potential energy and internal (thermal) energy,

$$E = \text{K.E.} + \text{P.E.} + U \quad (11)$$

or

$$e = \frac{1}{2}V^2 + gz + u \quad (12)$$

where  $u$  is the specific internal energy,  $V^2/2$  is the specific kinetic energy and  $gz$  is the specific potential energy.

## 5.2 First Law for a Control Volume

Consider a control volume  $\mathcal{V}$  bounded by control surface  $\mathcal{S}$ . The first law is essentially conservation of energy that states that *the rate of change of total energy inside the control volume must be balanced by the sum of the rate of work transfer into the control volume across the control surface, the rate of heat transfer into the control volume across the control surface, and rate of total energy transfer into the control volume by bulk flow across the control surface:*

$$\frac{d}{dt} \int_{\mathcal{V}} \rho e d\mathcal{V} + \int_{\mathcal{S}} \rho e \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} = \dot{Q}_{\mathcal{V}}^{\leftarrow} + \dot{W}_{\mathcal{V}}^{\leftarrow} \quad (13)$$

where

- the rate of change of total energy ( $E_{\mathcal{V}} = \int_{\mathcal{V}} \rho e d\mathcal{V}$ ) inside the control volume is  $\frac{dE_{\mathcal{V}}}{dt}$ ,
- the surface integral of the flux of total energy crossing the control surface is  $\dot{E}_{\mathcal{S}} = \int_{\mathcal{S}} \rho e \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S}$
- the rate of heat transfer across the control surface is  $\dot{Q}_{\mathcal{V}}^{\leftarrow}$ , and
- the rate of work done on the control volume is  $\dot{W}_{\mathcal{V}}^{\leftarrow}$

and where  $t$  is time, density is inverse of specific volume  $\rho \equiv \frac{1}{v}$ , specific total energy is the sum of specific internal energy  $u$ , specific kinetic energy and specific potential energy  $e \equiv u + \frac{1}{2}|\mathbf{u}|^2 + gz$ , where  $\mathbf{u}$  is the velocity vector<sup>2</sup>,  $z$  is the elevation and  $g$  is gravity. The term  $\hat{\mathbf{n}}$  is the unit normal vector on the control surface pointing away from the control volume.

## 6 The Flux Term

The flux of specific property  $b$  is nothing but the amount of  $\rho b$  carried across the control surface per unit area per unit time,  $\rho b \mathbf{u} \cdot \hat{\mathbf{n}}$ . Note when  $b = 1$ , we are then talking about the mass flux  $\rho \mathbf{u} \cdot \hat{\mathbf{n}}$ , with units of kg/m<sup>2</sup>/s. If  $b \equiv \mathbf{u}$ , then we are talking about the momentum flux,  $\rho \mathbf{u} \mathbf{u} \cdot \hat{\mathbf{n}}$  with units N/m<sup>2</sup>/s. And if  $b \equiv e$ , then we are talking about the total energy flux  $\rho e \mathbf{u} \cdot \hat{\mathbf{n}}$ , with units J/m<sup>2</sup>/s. Similarly the entropy flux is  $\rho s \mathbf{u} \cdot \hat{\mathbf{n}}$  with units J/K/m<sup>2</sup>/s, where  $s$  is the specific entropy.

The total mass crossing the boundary per unit time is the integral of the mass flux over the control surface,  $\dot{m}_{\mathcal{S}} = \int_{\mathcal{S}} \rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S}$ . The quantity  $\rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S}$  is the rate of mass crossing the differential surface element  $d\mathcal{S}$  and may be represented as  $dm = \rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S}$ . Similarly the total energy crossing the control surface is the integral of the total energy flux over the control surface  $\dot{E}_{\mathcal{S}} = \int_{\mathcal{S}} \rho e \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} = \int_{\mathcal{S}} e dm$ . Also the total entropy crossing the control surface is the integral of the entropy flux over the control surface

---

<sup>2</sup>Please note that  $u$  is the specific internal energy while bold  $\mathbf{u}$  is the velocity vector.

$$\dot{S}_S = \int_S \rho s \mathbf{u} \cdot \hat{\mathbf{n}} dS = \int_S s dm.$$

We now look at two special cases:

- At a solid boundary, the flux of any property is zero since the velocity component normal to the boundary is zero at the boundary; this is known as the no-through flow boundary condition at a solid boundary.
- If the enthalpy is uniform over a part of the control surface<sup>3</sup>, the rate of enthalpy transport out of the control volume across that part of control surface is  $\dot{m}h$ .

## 7 Energy Storage Modes: Internal Energy, Kinetic Energy, and Potential Energy

## 8 Energy Transfers Across Boundary of Control Volume/Mass

### 8.1 Work Interactions across Control Surface

The work done on the control volume may be split into three components

$$\dot{W}_{\mathcal{V}}^{\leftarrow} = \dot{W}_{\mathcal{V},\text{bulk}}^{\leftarrow} + \dot{W}_{\mathcal{V},\text{expansion}}^{\leftarrow} + \dot{W}_{\mathcal{V},\text{shaft}}^{\leftarrow} \quad (14)$$

where

- $\dot{W}_{\mathcal{V},\text{bulk}}^{\leftarrow} = - \int_S p \mathbf{u} \cdot \hat{\mathbf{n}} dS$  is the work done by bulk flow per unit time. This may be understood as follows. During the time period from  $t$  to  $t + dt$ , the amount of mass  $\delta m$  that crossed differential control surface area  $dS$  is  $\rho \mathbf{u} \cdot \hat{\mathbf{n}} dS dt$ . If we take a control mass that comprises of the control volume plus the  $\delta m$ , then at time  $t + dt$ , the amount of mass  $\delta m$  is now completely inside the control volume as shown in Figure 2. So we have essentially compression during which the volume of our control mass was reduced from  $\mathcal{V} + \delta V$  to  $\mathcal{V}$ , where  $\delta V = (\mathbf{u} \cdot \hat{\mathbf{n}} dS) dt$ . If  $p$  is the pressure of the differential mass  $\delta m$ , then the work done by bulk flow during time span  $dt$  on the control volume across differential surface element  $dS$  is  $\delta W^{\leftarrow} = -p \delta V = -p (\mathbf{u} \cdot \hat{\mathbf{n}} dS) dt$ . The rate of work done by bulk flow on the control volume across a differential control surface area  $dS$  is  $\frac{\delta W^{\leftarrow}}{\delta t} = -p \mathbf{u} \cdot \hat{\mathbf{n}} dS$ . Integrating over the control surface, we get

$$\dot{W}_{\mathcal{V},\text{bulk}}^{\leftarrow} = - \int_S p \mathbf{u} \cdot \hat{\mathbf{n}} dS \quad (15)$$

- If part of the control surface is moving such as the case of a moving piston, then we have essentially work done on control volume by expansion (or compression) according to

$$\dot{W}_{\mathcal{V},\text{expansion}}^{\leftarrow} = - \frac{d}{dt} \int_{\mathcal{V}} p^* d\mathcal{V} \quad (16)$$

where  $p^*$  is the pressure on the inner surface of the moving boundary.

---

<sup>3</sup>This part could correspond to an inlet or an outlet.

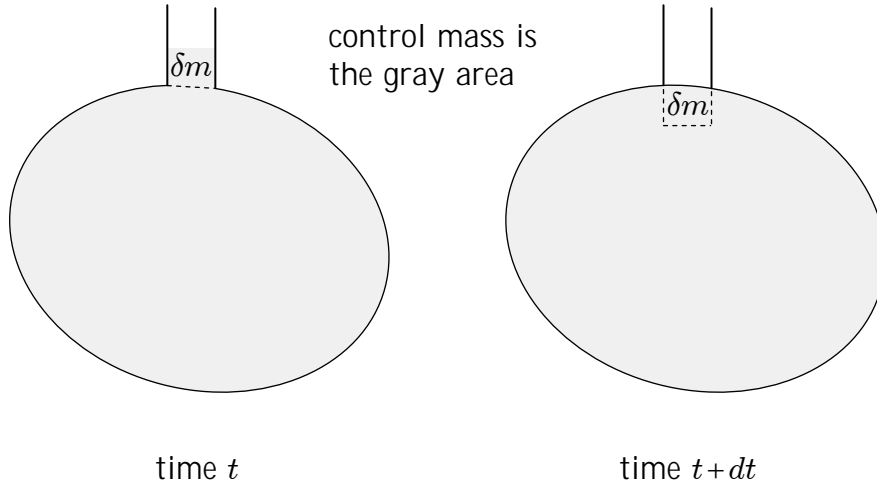


Figure 2: Schematic for understanding work done by bulk flow.

- Other forms of work crossing the control surface such as shaft work, electrical work are included in  $\dot{W}_{V,\text{shaft}}^{\leftarrow}$ .

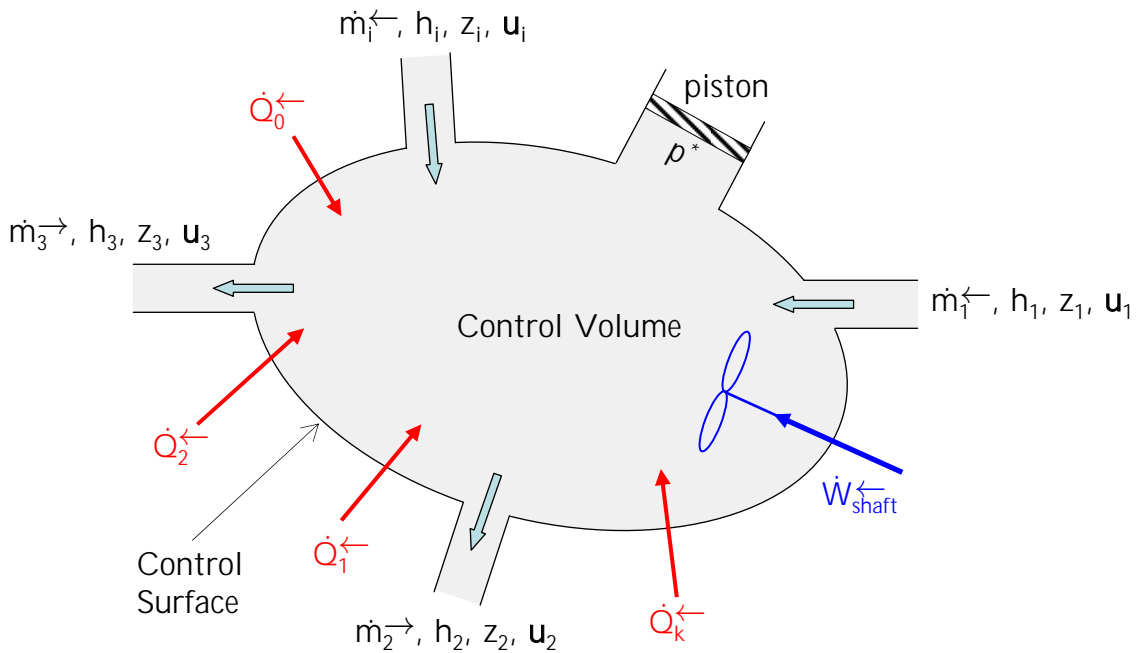


Figure 3: Schematic for conservation of energy for a control volume.

## 8.2 Heat Interactions across Control Surface

We assume that the control volume is exchanging heat  $\dot{Q}_0^{\leftarrow}$  with environment at temperature  $T_0$  in addition to other heat exchanges  $\dot{Q}_k^{\leftarrow}$  with various components or thermal reservoirs taking place at temperatures  $T_k$ , with  $k = 1, 2, \dots$ , so that

$$\dot{Q}^{\leftarrow} = \dot{Q}_0^{\leftarrow} + \sum_k \dot{Q}_k^{\leftarrow} \quad (17)$$



The reason we separated heat interaction with the environment from heat interaction with other systems is both useful for understanding the heat losses to the environments as well as for the second law formulation for a control volume.

## 9 Conservation Laws for a Control Volume

Noting the  $h = u + pv$  and substituting the expressions for heat and work in the first law, the conservation of mass and energy for a control volume

$$\frac{d}{dt} \int_{\mathcal{V}} \rho d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} = 0 \quad (18)$$

$$\frac{d}{dt} \int_{\mathcal{V}} \rho \left( u + \frac{1}{2} |\mathbf{u}|^2 + gz \right) d\mathcal{V} + \int_{\mathcal{S}} \rho \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right) \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} = \dot{Q}_0^{\leftarrow} + \sum_k \dot{Q}_k^{\leftarrow} - \frac{d}{dt} \int_{\mathcal{V}} p^* d\mathcal{V} + \dot{W}_{\mathcal{V}, \text{shaft}}^{\leftarrow} \quad (19)$$

## 10 Mutli Inlets/Outlets with Uniform Properties

We consider the case in which bulk flow takes places across multiple inlets and multiple outlets. We also assume that across the inlet/outlet cross sections (which is the intersection between the inlet/outlet and the control volume), the thermodynamic properties are uniform. In this case, the flux terms in the conservation of mass and energy equation becomes

$$\begin{aligned} \int_{\mathcal{S}} \rho \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} &= \sum_i \dot{m}_i^{\rightarrow} \\ \int_{\mathcal{S}} \rho \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right) \mathbf{u} \cdot \hat{\mathbf{n}} d\mathcal{S} &= \sum_i \dot{m}_i^{\rightarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i \end{aligned}$$

The conservation of mass and energy for a control volume can then be expressed

$$\frac{dm}{dt} = \sum_i \dot{m}_i^{\leftarrow} \quad (20)$$

$$\frac{dE}{dt} = \dot{Q}_0^{\leftarrow} + \sum_k \dot{Q}_k^{\leftarrow} - \frac{d}{dt} \int_{\mathcal{V}} p^* d\mathcal{V} + \dot{W}_{\mathcal{V}, \text{shaft}}^{\leftarrow} + \sum_i \dot{m}_i^{\leftarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i \quad (21)$$

where

$$\begin{aligned} m &= \int_{\mathcal{V}} \rho d\mathcal{V} \\ E &= \int_{\mathcal{V}} \rho \left( u + \frac{1}{2} |\mathbf{u}|^2 + gz \right) d\mathcal{V} \end{aligned}$$

## 11 Special Cases

### 11.1 Steady State

If, in addition to the assumption that each inlet or outlet is characterized by uniform properties across its cross section, the control volume is in a steady state, then  $d()/dt = 0$ ,

then

$$\sum_i \dot{m}_i^{\leftarrow} = 0 \quad (22)$$

$$\dot{Q}_0^{\leftarrow} + \sum_k \dot{Q}_k^{\leftarrow} + \dot{W}_{\mathcal{V},\text{shaft}}^{\leftarrow} + \sum_i \dot{m}_i^{\leftarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i = 0 \quad (23)$$

## 11.2 Unsteady Process

If the control volume undergoes an unsteady process between  $t_1$  and  $t_2$ , then integrating equations (20) and (21) from  $t_1$  to  $t_2$ , we get

$$(m_2 - m_1)_{\mathcal{V}} = \sum_i m_i^{\leftarrow} \quad (24)$$

$$(E_2 - E_1)_{\mathcal{V}} = Q_0^{\leftarrow} + \sum_k Q_k^{\leftarrow} - \int_{\mathcal{V}_1}^{\mathcal{V}_2} p^* d\mathcal{V} + W_{\text{shaft}}^{\leftarrow} + \sum_i \int_{t_1}^{t_2} \dot{m}_i^{\leftarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i dt \quad (25)$$

where

$$\begin{aligned} m_1 &= \int_{\mathcal{V}_1} \rho d\mathcal{V} \\ m_2 &= \int_{\mathcal{V}_2} \rho d\mathcal{V} \\ m_i^{\leftarrow} &= \int_{t_1}^{t_2} \dot{m}_i^{\leftarrow} dt \\ E_1 &= \int_{\mathcal{V}_1} \rho e d\mathcal{V} \\ E_2 &= \int_{\mathcal{V}_2} \rho e d\mathcal{V} \\ Q_0^{\leftarrow} &= \int_{t_1}^{t_2} \dot{Q}_0 dt \\ Q_k^{\leftarrow} &= \int_{t_1}^{t_2} \dot{Q}_k dt \\ W_{\text{shaft}}^{\leftarrow} &= \int_{t_1}^{t_2} \dot{W}_{\text{shaft}}^{\leftarrow} dt \end{aligned}$$

If the thermodynamic properties are spatially uniform inside the control volume, then

$$(E_2 - E_1)_{\mathcal{V}} = (m_2 e_2 - m_1 e_1)_{\mathcal{V}} \quad (26)$$

If for each inlet/outlet the thermodynamic properties are constant (spatially uniform and time independent) across the cross section, then

$$\sum_i \int_{t_1}^{t_2} \dot{m}_i^{\leftarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i dt = \sum_i m_i^{\leftarrow} \left( h + \frac{1}{2} |\mathbf{u}|^2 + gz \right)_i \quad (27)$$