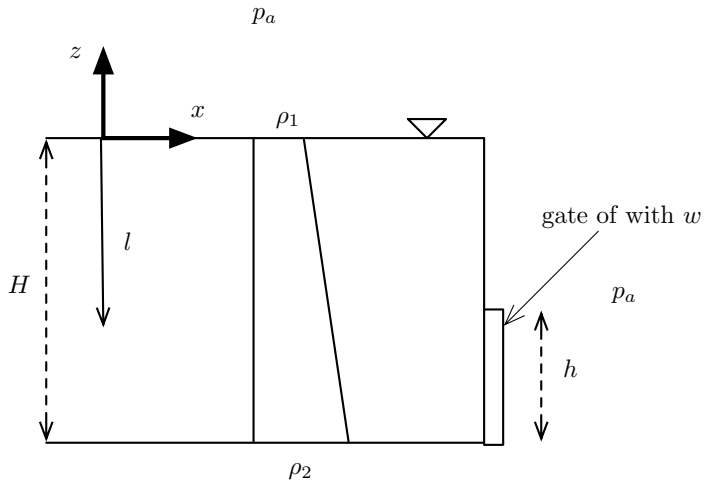
Drop Quiz 2

Given a body of fluid with density that varies linearly from $\rho_1$at the surface to ρ_2 at the bottom (depth H). What is the force due to pressure acting on a rectangular vertical gate of height h (from depth $H-h$ to depth H). The width of the gate is w.



Solution
The equation governing fluid statics is

$$-\nabla p + \rho \mathbf{g} = \mathbf{0}$$

Noting that $\mathbf{g} = -g\hat{\mathbf{z}}$, then

$$\frac{dp}{dz} = -\rho g$$

Since $l = -z$ (see figure), then $p - p_a = \int_0^l \rho g \, dl$.

Since the density varies linearly from ρ_1 at the surface to ρ_2 at the bottom then $\rho = \rho_1 + \frac{l}{L}(\rho_2 - \rho_1)$. Then

$$p(l) - p_a = \int_0^l \left[\rho_1 + \frac{l}{L}(\rho_2 - \rho_1)\right] g \, dl = g\left(\rho_1 l + \frac{l^2}{2L}(\rho_2 - \rho_1)\right)$$

The force due to pressure is

$$\begin{aligned} \mathbf{F} &= \int -p\hat{\mathbf{n}} \, dA = \hat{\mathbf{x}} w \int_{H-h}^{H} (p - p_a) \, dl \\ &= \hat{\mathbf{x}} w g \int_{H-h}^{H} \left(\rho_1 l + \frac{l^2}{2L}(\rho_2 - \rho_1)\right) dl \end{aligned}$$