Quiz 2

This is a closed book exam. You have 90 minutes to work on this exam. Good Luck !
Please work independently. Your work should be neat and your handwriting should be clear.
Read the whole exam before you start.

Problem 1 - High Rise - 30 pts

A Mechanical Engineer is interested in investigating the moment induced by areodynamic drag due to flow of wind around a model of a high-rise bluiding in a wind tunnel. The test is conducted at a large Reynolds Number so that viscous effects may be neglected.
(a) Using dimensional analysis, relate the moment T of the aerodynamic drag about the base of the building to the velocity gradient dV/dz, building height H, and air density ρ.
(b) A model test is run for $(dV/dz)_m = 1s^{-1}$, $H_m = 1m$ and $\rho_m = 1.25$ kg/m^3. The measured value of $T_m = 0.03Nm.$, what would be the numerical value of the prototype T_p if $H_p = 100m$, $\rho_p = 1.25kg/m^3$ and $(dV/dz)_p = 0.30s^{-1}$?
(c) Using dimensional analysis, express the form of the functional relationship between the pressure $p(z) - p_a$ and dV/dz, H, ρ, and the height z above the base.

Problem 2 - Bechtel 3rd floor - 30 pts
A pump delivers a volume flow-rate Q (m^3/s) of water through a pipe in the water fountain shown in the Figure. If H is the distance between the reservoir and the highest point of the jet exiting the orifice, what should the gauge pressure p_r be in the reservoir ? What should the diameter d of the orifice be ? Given:

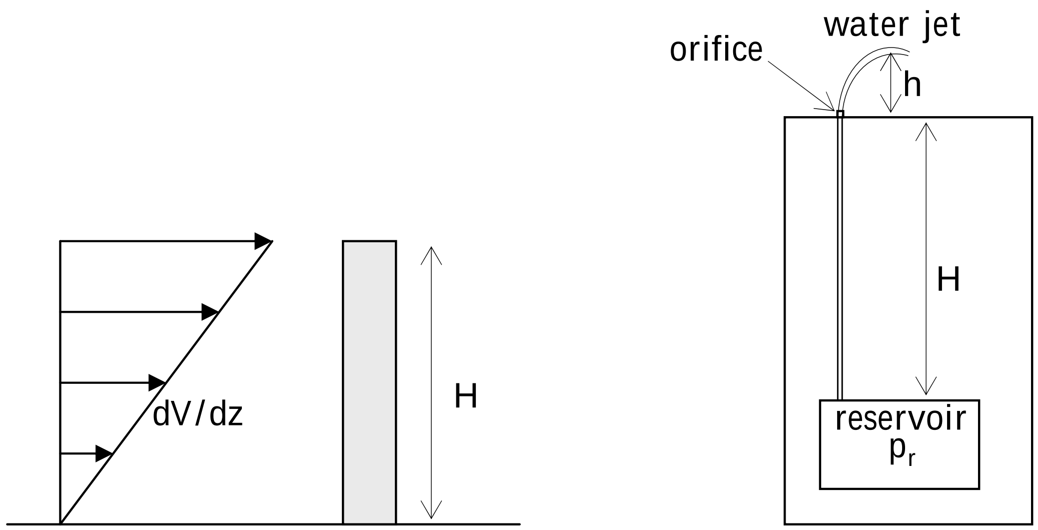

Figure 1: Schematics of Problem 1 (left) and Problem 2 (right).

Problem 1 Solution - High Rise

(a) $T = f(dV/dz, H, \rho)$. The units of the various variables are:

$$\begin{aligned}[T] &= \mathcal{M}\mathcal{L}^2\mathcal{T}^{-2} \\ [dV/dz] &= \mathcal{T}^{-1} \\ [H] &= \mathcal{L} \\ [\rho] &= \mathcal{M}\mathcal{L}^{-3}\end{aligned}$$

The number of dimensionless variables is 4-3 = 1. So that

$$\begin{aligned}\Pi &= [T]\,[dV/dz]^{\alpha}\,[H]^{\beta}\,[\rho]^{\gamma} = 1 \\ &= (\mathcal{M}\mathcal{L}^2\mathcal{T}^{-2})(\mathcal{T}^{-1})^{\alpha}(\mathcal{L})^{\beta}(\mathcal{M}\mathcal{L}^{-3})^{\gamma} = 1\end{aligned}$$

Leading to the following equations

$$\begin{aligned}&1 + \gamma = 0 \\ &2 + \beta - 3\gamma = 0 \\ &-2 - \alpha = 0\end{aligned}$$

We get $\gamma = -1$, $\alpha = -2$, and $\beta = -5$ so that

$$\Pi = \frac{T}{\rho H^5 (dV/dz)^2} = \text{constant}$$

(b) Dynamic similarity requires

$$\left(\frac{T}{\rho H^5 (dV/dz)^2}\right)_m = \left(\frac{T}{\rho H^5 (dV/dz)^2}\right)_p$$

so that

$$T_p = T_m \frac{(\rho H^5 (dV/dz)^2)_p}{(\rho H^5 (dV/dz)^2)_m} = 0.03 \times (100^5) \times (0.3^2) \times (1) = 27 \times 10^6 \text{N m}$$

(c) $p(z) - p_z = f(dV/dz, H, \rho, z)$. The units of the various variables are:

$$\begin{aligned}[p(z) - p_a] &= \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-2} \\ [dV/dz] &= \mathcal{T}^{-1} \\ [H] &= \mathcal{L} \\ [z] &= \mathcal{L} \\ [\rho] &= \mathcal{M}\mathcal{L}^{-3}\end{aligned}$$

resulting in

$$\frac{p(z) - p_a}{\rho (dV/dz)^2 H^2} = f\left(\frac{z}{H}\right)$$

Problem 2 Solution

(a) Applying Bernoulli's equation along a streamline starting from (1) inside the reservoir all the way to (2) the highest point in the jet and noting that $V_1 \simeq 0$, $p_1 = p_r$, $V_2 = 0$, and $p_2 = p_a$, then

$$p_r - p_a = \rho g(H + h)$$

(b) Applying Bernoulli's equation along a streamline starting from (1) the exit at the orifice to (2) the highest point in the jet and noting that $p_1 = p_a$, then

$$\frac{1}{2}\rho V_1^2 = \rho g h \Rightarrow V_1 = \sqrt{\rho g h}$$

The diameter d of the orifice should then be

$$Q = \frac{\pi d^2}{4} V_1 \Rightarrow d = \sqrt{\frac{4Q}{\pi V_1}} = \sqrt{\frac{4Q}{\pi \sqrt{\rho g h}}}$$

Problem 3 - Easy Money - $40

D.C. stumbled upon a huge oil well in Neverland. Soon after, he founded Hapybolton Inc. for selling and pumping crude oil of density ρ_o and viscosity μ_o to nearby nations. His partner, D.R. proposed to mix the oil with sea water of density ρ_w and viscosity μ_w to increase the profit. One of the pipes (of length L) is to carry a volume flow-rate Q from a station in Neverland to Tapline station in Zahrani. Unfortunately for Hapybolton, the supervising engineer in Tapline, who happened to be an AUB graduate and particularly knowledgeable in Fluid Mechanics, was able to discover the foul play by simply reading the pressure difference Δp between the inlet and exit of the pipe. Because sea water is lighter than crude oil and since the two fluids are immiscible, the water layer simply floats on top of the oil layer (of height H_o) as shown in the Figure below. To simplify the problem, we assume that the pipe is of rectangular cross section of height H and width W, with $W >> H$ so that flow may be modeled as two dimensional with $u = u(y)$. In this problem, gravitational effects are neglected and the flow is assumed to be fully developed.
(a) What are the conditions at the water-oil interface?
(b) What are the boundary conditions at the lower and upper walls of the channel?
(c) What are the equations governing the velocity in the oil and water sections of the channel?
(d) Relate the volume flow-rate Q to the pressure difference Δp, channel length L, width W, height H, oil and water viscosities μ_o and μ_w.
(e) For the case of only oil and the case of water-oil, compare the pressure difference required to establish the desired volume flow-rate Q. Which Δp is larger, why?

[*M.H.S. certified*]

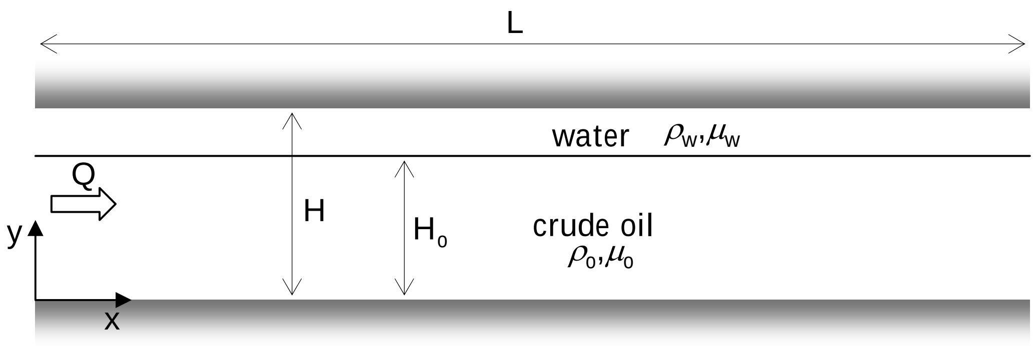

Figure 2: Schematic of Problem 3.

Problem 3 Solution

(a) At the oil-water interface, we must have continuity of velocity and of shear so that

$$u_o(y = H_0) = u_w(y = H_0)$$
$$\mu_o \left(\frac{\partial u_o}{\partial y}\right)_{y=H_0} = \mu_w \left(\frac{\partial u_w}{\partial y}\right)_{y=H_0}$$

(b) the no-slip conditions at the solid boundary are

$$u_o(y = 0) = 0$$
$$u_w(y = H) = 0$$

(c) With gravity neglected, the $y-$component of the momentum equation leads to $p = p(x)$ so that the x-component of the momentum equation for both warer and oil are

$$\text{for water} \quad \mu_w \frac{\partial^2 u_w}{\partial y^2} = \frac{dp}{dx}$$
$$\text{for oil} \quad \mu_o \frac{\partial^2 u_o}{\partial y^2} = \frac{dp}{dx}$$

(d) Solving the above equations lead to

$$\mu_w \frac{\partial u_w}{\partial y} = \frac{dp}{dx} y + C_{w1} \Rightarrow u_w = \frac{1}{2\mu_w}\frac{dp}{dx} y^2 + \frac{C_{w1}}{\mu_w} y + C_{w2}$$
$$\mu_o \frac{\partial u_o}{\partial y} = \frac{dp}{dx} y + C_{o1} \Rightarrow u_o = \frac{1}{2\mu_o}\frac{dp}{dx} y^2 + \frac{C_{o1}}{\mu_o} y + C_{o2}$$

If we choose our reference frame at the interface, the no-slip boundary conditions result in

$$\frac{1}{2\mu_o}\frac{dp}{dx} H_o^2 - \frac{C_{o1}}{\mu_o} H_o + C_{o2} = 0$$
$$\frac{1}{2\mu_w}\frac{dp}{dx} H_w^2 + \frac{C_{w1}}{\mu_w} H_w + C_{w2} = 0$$

where $H_w = H - H_o$. The conditions at the interface lead to

$$C_{o1} = C_{w1} = C_1$$
$$C_{o2} = C_{w2} = C_2$$

Solving for C_1 and C_2 yields

$$C_1 = -\frac{1}{2}\frac{dp}{dx}\left(\frac{H_w^2}{\mu_w} - \frac{H_o^2}{\mu_o}\right)\left(\frac{H_w}{\mu_w} + \frac{H_o}{\mu_o}\right)^{-1}$$
$$C_2 = -\frac{1}{2}\frac{dp}{dx}(H_o + H_w)\left(\frac{H_w}{\mu_w} + \frac{H_o}{\mu_o}\right)^{-1}$$

The total volume flow rate per unit depth is

$$Q = Q_o + Q_w = \int_{-H_0}^{0} u_o\, dy + \int_0^{H_w} u_w dy$$

(e) If $\mu_o > \mu_w$, then oil presents more friction than oil plus water for the same volume flow rate so that a larger $\Delta p = p_{in} - p_{out}$ is required for the oil only case. Note that ρ (i.e. interia) plays no role here because in steady state the acceleration term is zero.