MECH 314 Introduction to Fluid Mechanics **Quiz 1** **April 1, 2009**
90 minutes

Hello people! Please read the questions carefully. As you solve them, indicate clearly your system or CV boundaries and any assumptions you might make. When possible, please state in words what you are trying to solve so that we can follow your work and give you credit. This is especially important if you run out of time. Keep track of time; if you get stuck, move on. At the end of the quiz, return the question sheet with the solution booklet. This is a closed book, closed neighbor quiz. One A4 sheet of your own notes is allowed. MOST IMPORTANTLY: relax and have a good time.

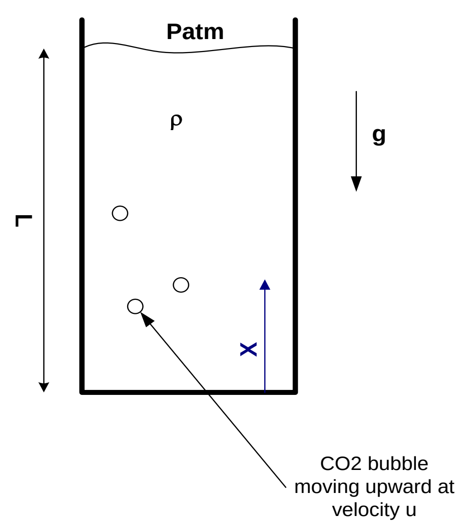
Problem 1 [50 points]

One day, before his death from lung cancer, the Marlboro Man noticed carbon dioxide bubbles rising up in his glass of Pepsi. He noticed, in particular, that some bubbles travelled faster than others. Let us analyze this situation.

First, assume that the shape of the bubbles is spherical. We know that a sphere traveling at a velocity u in a fluid with density ρ, will experience a drag force, D, opposite to the direction of motion: $D = \pi\rho d^2u^2/32g$. Where ρ = fluid density, d = sphere diameter, u = velocity of sphere, g = gravity constant.

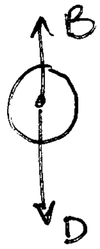
Furthermore, we will assume that the bubbles consist of carbon dioxide, with gas constant R, which behaves as an ideal gas (PV=mRT). We will also assume that the temperature in the bubble is constant, that surface tension is negligible, and that the mass of gas in the bubble remains constant.

a) (15 pts) Do large or small bubbles travel faster? Derive an expression for u as a function of ρ, g, and d, neglecting the inertia and weight of the gas in the bubble.
b) (15 pts) What happens to the size of a bubble as it travels upward? Derive an expression for d(x) as a function of its position x relative to the glass bottom, given that the initial bubble size was d_o at x = 0.
c) (10 pts) Using reasonable numbers for ρ, L, and P_{atm} by what percent will the diameter change as a bubble travels from the bottom to the top of the glass?
d) (10 pts) Taking into account your result for c), derive an expression for the time it takes for a bubble of initial diameter d_o to travel from the bottom to the top of the glass.



a) do large or small bubbles travel faster?

· take bubble as a system



neglect inertia

+S $\sum F_{bubble} = \cancel{ma} = B - D$ (neglect weight of bubble)

where: buoyancy is B

drag is D

$\Rightarrow \sum F = 0 = B - D$ OR $\boxed{B = D}$

B = weight of displaced Pepsi

$= \frac{\pi}{6} d^3 \rho_{Pepsi}\, g$

$D = \pi \rho_{Pepsi} d^2 u^2 / 32g$

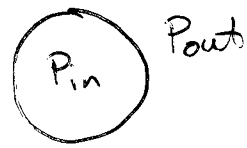
$\frac{\pi}{6} d^3 \rho_p g = \frac{\pi \rho d^2 u^2}{32g}$

$\Rightarrow u = \left(\frac{16}{3} g^2 d\right)^{1/2}$

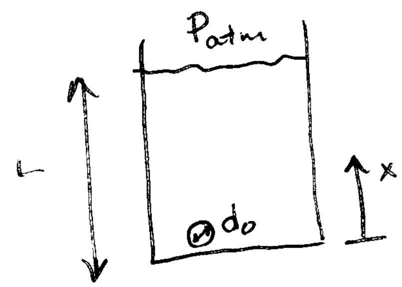
larger bubbles travel faster because the drag increases with d^2 while the buoyancy increases with d^3

b) what happens to size of bubble as it travels up?

- since surface tension is neglected, $P_{in} = P_{out}$ (given)



- P_{out} follows a hydrostatic pressure condition:



$$P(x) = P_{atm} + \rho_{Pepsi}\, g\,(L-x)$$

- CO_2 is treated as an ideal gas:

$PV = mRT$ but m, T are constant (given)

$\Rightarrow PV = \text{const.}$

$P_0 V_0 = PV$ where P_0 is pressure @ $x=0$, V_0 "volume"

$\Rightarrow V = \frac{P_0}{P} V_0$ OR $d^3 = \frac{P_0}{P} d_0^3$ ($\pi/6$ cancels)

where $P_0 = P_{atm} + \rho_{Pepsi}\, g\, L$

$P = P_{atm} + \rho_{Pepsi}\, g\,(L-x)$

$$\Rightarrow d = d_0 \left[\frac{P_{atm} + \rho_{Pepsi}\, g\, L}{P_{atm} + \rho_{Pepsi}\, g\,(L-x)} \right]^{1/3}$$

as x increases, so does d

quick check: what is d when $x = 0$?

c) reasonable numbers

$L = 20\,cm = 0.2\,m$

$\rho_{Pepsi} = \rho_{water} = 1000\,Kg/m^3$

$P_{atm} = 101 \times 10^3\ \frac{N}{m^2}$

then $\frac{d}{d_o} =$

i.e. it practically does not change at all!

$\Rightarrow$ can treat d as constant in a glass of Pepsi

d) since $d \sim$ const

then $u = \left(\frac{16}{3} g^2 d_o\right)^{1/2} = \frac{dx}{dt} = \text{const}$

$$\int_0^L dx = \int_0^{t_{final}} u\,dt = u\,t_{final}$$

$$\boxed{t = \frac{L}{u} = \frac{L}{\left(\frac{16}{3} g^2 d_o\right)^{1/2}}}$$

Grading guidelines for Problem 1

Part (a)

- 5 points for recognizing that the sum of forces = 0
- 8 points for calculating the buoyancy force
- 2 points for the final answer and your interpretation of it

Part (b)

- 4 points $P_{outside\ bubble} = P_{inside\ bubble}$ since surface tension is negligible (given)
- 4 points for correct hydrostatic pressure equation
- 4 points for recognizing PV = const
- 3 points for final answer

Part (c)

- 2 points for reasonable L
- 2 points for reasonable rho
- 1 point for reasonable Patm
- 5 points for drawing the correct conclusion based on your calculation

Part (d) (0,5,or 10 points)

- since d approx constant, u is approx constant
- then t = L/u
- final answer correct

Quiz 1

Problem 2 [50 points] (a) [20 pts] , (b) [10 pts], (c) [20 pts], bonus: (d) [3 pts], (e) [3 pts]

A bucket of mass m_b rests on a scale with a spring constant k and unstressed length y_0. A jet of water of diameter D_j falls toward the bucket at a steady velocity V (relative to the ground). The cross sectional area of the bucket is A. The density of water is ρ. Initially there is no water in the bucket. Assume that, except for the rise of water in the bucket, the velocity of water in the bucket relative to the bucket is zero.
(a) If the spring is rigid (stiffness is infinite), what is the force on the spring $F_o(t)$ (as the bucket is filling) in terms of given quantities ?

In the following part, the spring has a finite stiffness and obeys the classical linear relashionship between force and change in length.
(b) Derive an expression for the water level of the bucket $h(t)$ and the mass of water in the bucket $m_w(t)$ as the bucket is moving in terms of $y(t)$ and other given quantities.
(c) Derive an expression for force on the spring in terms of $y(t)$ and other given quantities. [you are NOT required to solve the differential equation.]
(d) What would the force measured by the spring be if we neglect the dynamics of motion of the bucket (i.e. neglect time derivatives of $y(t)$).
(e) Approximate the solution you got in part (b) after a long time has passed and for $A_j/A << 1$, where A_j is jet cross sectional area. (assume overflow has not yet occured.)

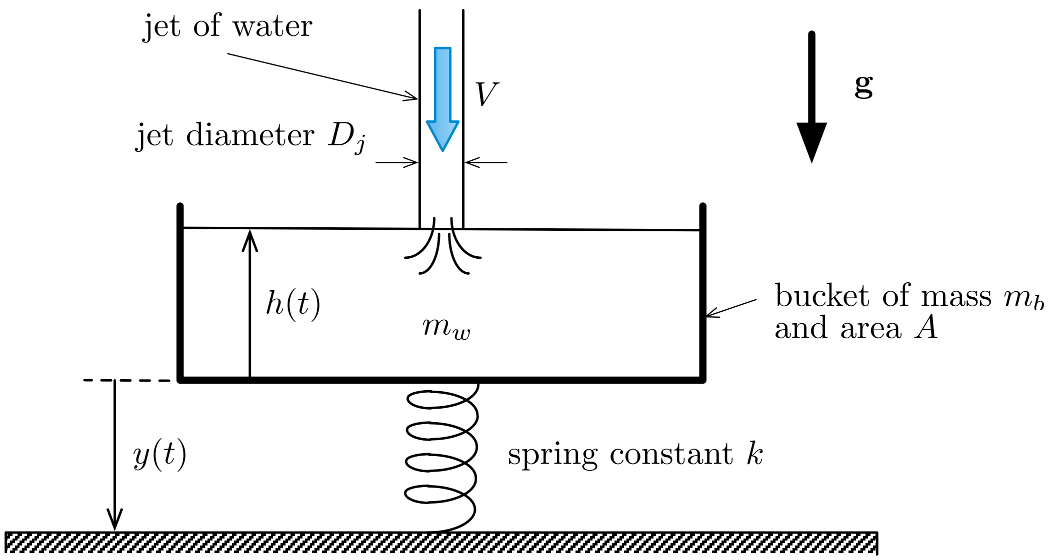


Figure 1: Schematic for problem 2.

Problem 1 Solution

(a) Choosing a control volume (See Fig. 2) where the upper control surface rises with water level.

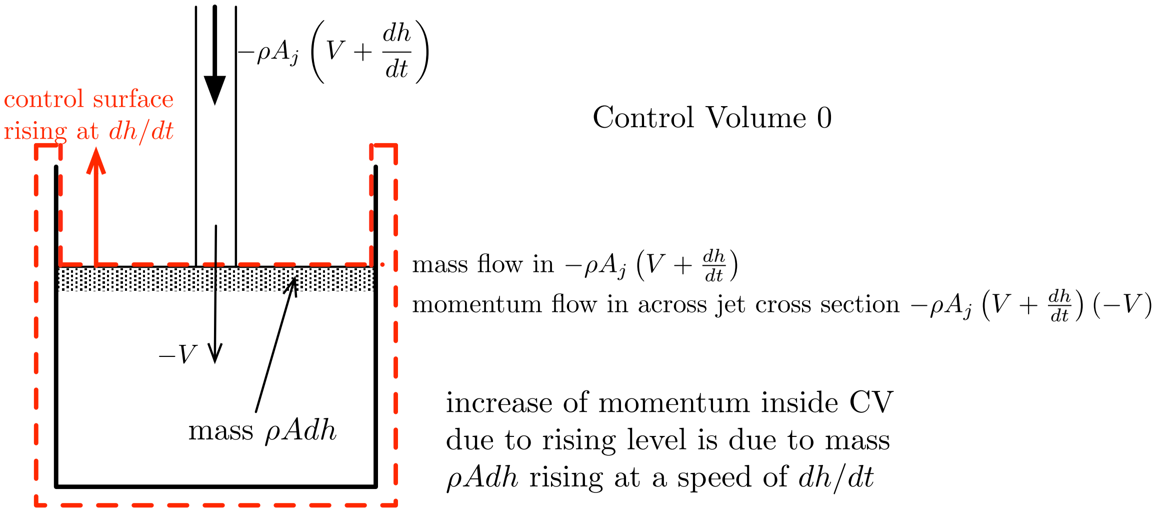


Figure 2: Control volumes.

Conservation of mass:

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u}_r \cdot \hat{\mathbf{n}} \, d\mathcal{S}$$

The volume of the control volume is $Ah(t)$ and the relative inflow velocity of the jet is $-(V + dh/dt)$, then

$$\rho A \frac{dh}{dt} - \rho \left(V + \frac{dh}{dt}\right) A_j = 0 \Rightarrow \frac{dh}{dt} = V \frac{A_j}{A - A_j}$$

$$\Rightarrow \quad h = \frac{A_j}{A - A_j} V t$$

Conservation of momentum, in the z direction

$$\sum F_z = \frac{\partial}{\partial t}\int_{\mathcal{V}} \rho u_z \, d\mathcal{V} + \int_{\mathcal{S}} \rho u_z (\mathbf{u}_r \cdot \hat{\mathbf{n}}) \, d\mathcal{S}$$

The forces are the reaction force of the spring R_s, weight of bucket $-m_b g$, and weight of water in bucket $-\rho A h g$. As for the transient term, note that except for the rising water, the velocity of water inside the bucket relative to the bucket is zero. So we divide the control volume into two parts: first part of volume $A(h - dh)$ with zero velocity, and a second part of volume $d\mathcal{V}^\star = A\,dh$ rising at a speed of $V^\star = dh/dt$. Then

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho u_z \, d\mathcal{V} = \rho V^\star \frac{d\mathcal{V}^\star}{dt} = \rho A \left(\frac{dh}{dt}\right)^2$$

Regarding the flux term, the inflow velocity in the chosen reference frame is $u_z = -V$ and the mass flux component is $\mathbf{u}_r \cdot \hat{\mathbf{n}} = -(V + dh/dt)$, so that the rate of momentum entering the control volume is $\rho A_j V(V + dh/dt)$. The momentum equation now looks like

$$R_s - (m_b + \rho A h)g = \rho A \left(\frac{dh}{dt}\right)^2 + \rho A_j V \left(V + \frac{dh}{dt}\right)$$

Then

$$R_s = (m_b + \alpha \rho A V t)g + \rho \frac{\alpha^2 A^2}{A_j} V^2$$

where $\alpha = A_j/(A - A_j)$.

(a1) Work through conservation of mass and momentum by choosing control volume 1 (See Fig. 3) consisting of the bucket and the water in the bucket such that rising water level is inside the control volume.

(a2) Now we solve part (a) by choosing a control volume 2 (See Fig. 3) consisting of the bucket and the water in the bucket, with upper control surface <u>fixed</u>. In this case the transient terms in the conservation of mass and momentum disappear. There are however extra terms due to the mass and momentum fluxes crossing the upper control surface.

Note that velocity of the flow across the jet cross section of the control surface is $-(V + dh/dt)$; the jet is going down with V and the water is rising with dh/dt. So for conservation of mass, there are two mass fluxes, an inflow due to the jet $-\rho A_j \left(V + \frac{dh}{dt}\right)$ and an outflow through the upper control surface $\rho A(dh/dt)$:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u}_r \cdot \hat{\mathbf{n}} \, d\mathcal{S} \\ \Rightarrow \quad & 0 - \rho A_j \left(V + \frac{dh}{dt}\right) + \rho A \frac{dh}{dt} = 0 \\ \Rightarrow \quad & \frac{dh}{dt} = V \frac{A_j}{A - A_j} \end{aligned}$$

As for the conservation of momentum, the momentum carried by the jet inflow is $-\rho A_j \left(V + dh/dt\right)(-V)$ and the momentum carried out by the rising water level is $\rho A (dh/dt)(dh/dt)$, so that

$$R_s - (m_b + \rho A h)g = 0 - \rho A_j \left(V + \frac{dh}{dt}\right)(-V) + \rho A \left(\frac{dh}{dt}\right)^2$$

which is the same result we got in part (a).

(b) Choosing control volume 1 (See Fig. 3) in a frame of reference attached to the bucket

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \, d\mathcal{V} + \int_{\mathcal{S}} \rho \mathbf{u}_r \cdot \hat{\mathbf{n}} \, d\mathcal{S} = 0$$

The jet velocity relative to the control surface, now moving at speed $\dot{y}$, is $-\rho A_j \left(V + \dot{y} + dh/dt\right)$. Notice that the way y is shown in the figure, $\dot{y}$ is already negative since the $y(t)$ decreases as

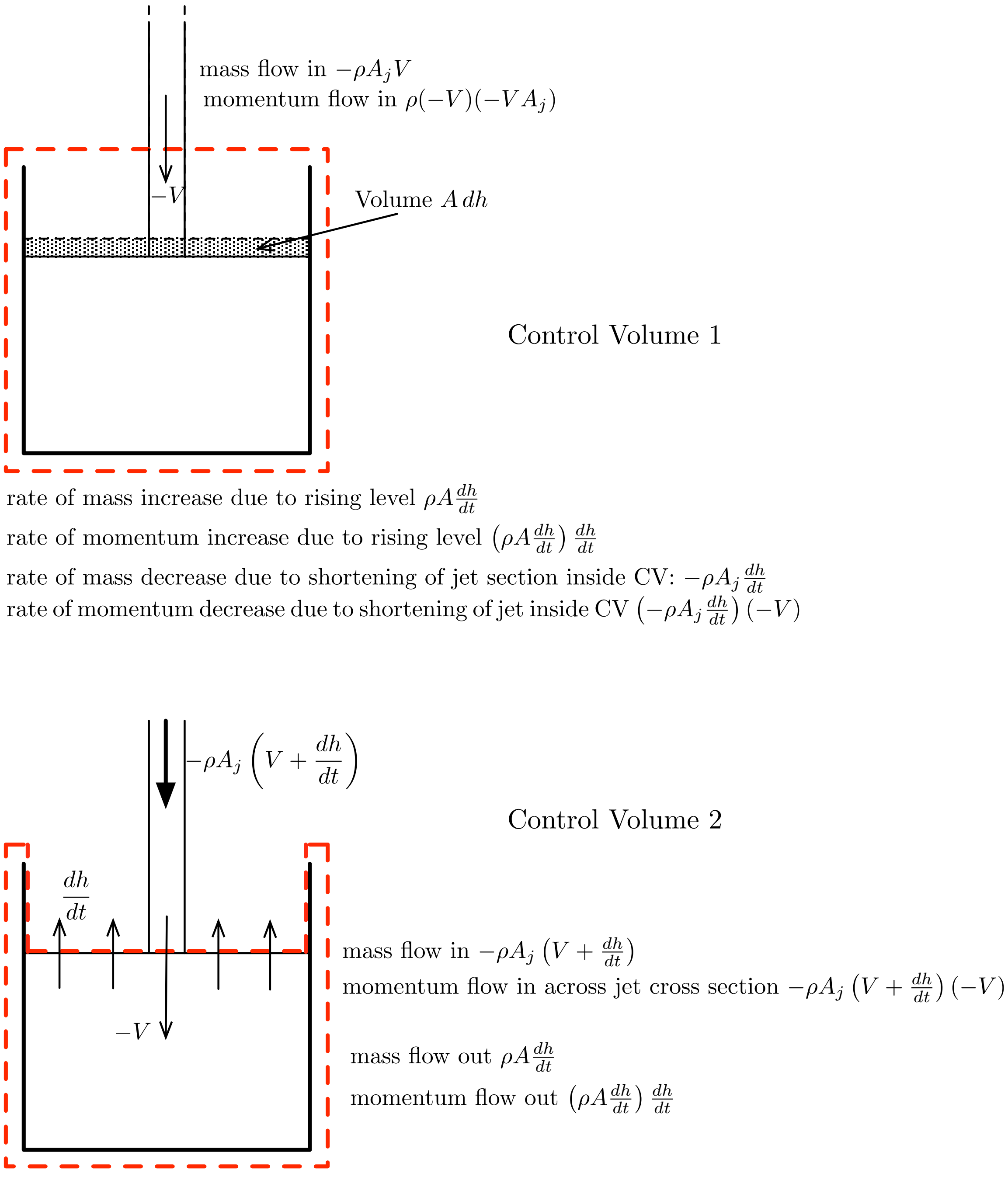


Figure 3: Control volumes.

the jet moves downwards. Noting that the upper control surface rises with the water level $h(t)$ (now defined in the moving reference frame), then conservation of mass yields

$$\rho A \frac{dh}{dt} - \rho A_j \left(V + \dot{y} + \frac{dh}{dt} \right) = 0 \Rightarrow \frac{dh}{dt} = \frac{A_j}{A - A_j} (V + \dot{y})$$

and the mass of water inside the bucket is given by

$$\frac{dm_w}{dt} = \rho A \frac{dh}{dt}$$
$$\Rightarrow \quad m_w = \rho A \frac{A_j}{A - A_j} (Vt + (y - y_0))$$

where $y_0 = y(t = 0)$.

(c) Applying conservation of momentum in a frame of reference (which is a non-inertial frame of reference) moving with the bucket. Choosing control volume 0 (See Fig. 2) and taking only the vertical component of the momentum equation,

$$\sum F_y - (m_b + m_w) \frac{d^2 y}{dt^2} = \frac{\partial}{\partial t} \int_{\mathcal{V}} \rho u_z \, d\mathcal{V} + \int_{\mathcal{S}} \rho u_z (\mathbf{u}_r \cdot \hat{\mathbf{n}}) \, d\mathcal{S}$$

We note the following

- Note that the forces in the vertical direction are the weight $-(m_b + m_w)g$ and the spring force $k(y_0 - y)$.
- The transient term is due to mass $\rho A \, dh$ rising at speed dh/dt in the moving reference frame.
- The rate of momentum crossing the control surface is due to jet is $-\rho A_j \left(V + \dot{y} + \frac{dh}{dt} \right) [-(V + \dot{y})]$, where $-(V + \dot{y})$ is the flow velocity in the moving reference frame.

The conservation of momentum becomes

$$k(y_0 - y) = (m_b + \alpha \rho A (Vt + (y - y_0))) (\ddot{y} + g) + \rho \frac{A^2 \alpha^2}{A_j} (V + \dot{y})^2$$

(d) If we neglect the time variation of y, the equation is

$$k(y_0 - y) = (m_b + \rho A_j (Vt + (y - y_0))) \, g + \rho \frac{A^2 \alpha^2}{A_j} V^2$$

(e) For long times $Vt >> y_0 - y$, $\rho A_j Vt >> m_b$, and for $A_j / A << 1$, we get

$$k(y_0 - y) = \rho A_j V t (\ddot{y} + g) + \rho A_j (V + \dot{y})^2$$