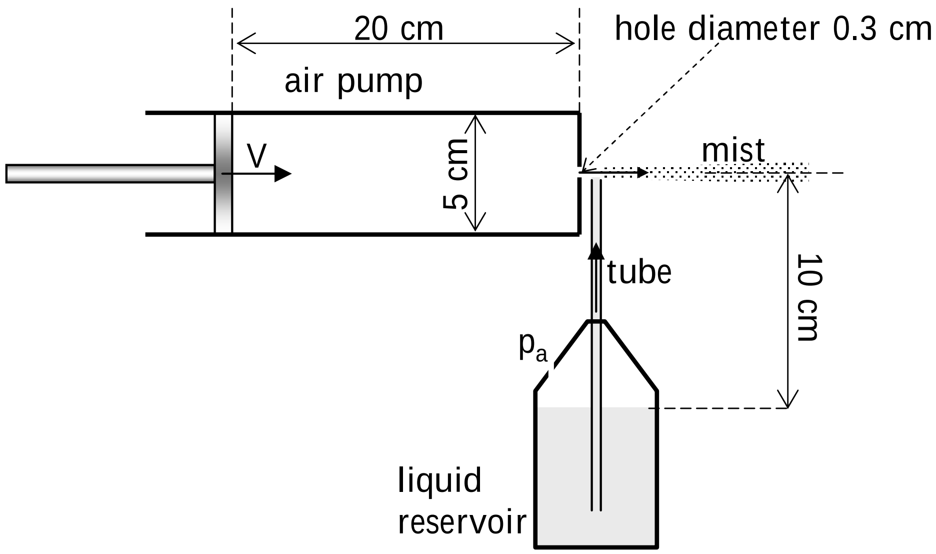
Quiz 2

- This is a closed book exam. You are allowed to bring in two A4 cheat sheets (4 pages.)
- Please write down the name of your instructor.
- You have 90 minutes.

Problem 1 (34 points)
An atomizer is made by connecting a simple hand held air pump to a liquid reservoir by a short tube as shown in Figure. The pump forces air of high velocity through the hole (of diameter 0.3 cm) which drives the liquid up through the tube to generate a fine mist of paint or pesticide. The liquid reservoir is open to the atmosphere and the vertical distance between its level and the hole is $h = 10$ cm. The bore (diameter) and stroke of the pump are 5 cm and 20 cm respectively.
(a) If the velocity of the piston is V, What is the velocity V_e at pump exit?
(b) What should the pressure at the pump exit be to initiate atomization (i.e. suction of the liquid)?
(c) What is the piston velocity required to initiate atomization?



Problem 1 Solution

(a) Conservation of mass: $\rho V A_p = \rho V_e A_e \Rightarrow V_e = V\frac{A_p}{A_e}$ where A_p and A_e are respectively the piston and exit hole cross sectional areas.

(b) Bernoulli along a streamline in the liquid from the surface of the liquid inside the reservoir to the exit of the tube at the hole exit:

$$p_a + \frac{1}{2}\rho_l V_1^2 + \rho_l g z_1 = p_e + \frac{1}{2}\rho_l V_e^{*2} + \rho_l g z_e$$

where $z_e - z_1 = h$ and we set $V_e^* = 0$ and $V_1 = 0$ so that the pressure required to initiate atomization is

$$p_e = p_a - \rho_l g h$$

where ρ_l is the density of the liquid.

(c) Bernoulli along a streamline in the jet from a point just ahead of the piston to a point in the jet right at the exit

$$p_1 + \frac{1}{2}\rho_a V = p_e + \frac{1}{2}\rho_a V_e^2$$

Noting that $p_1 = p_a$ (sum of forces acting on the frictionless piston is equal to zero since its speed is constant) and $p_j = p_e$ then

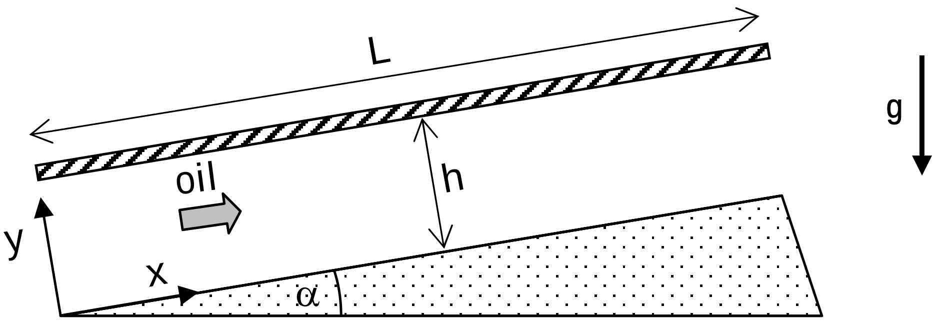
$$V^2 = V_j^2 - \frac{2(p_a - p_e)}{\rho_a} = V^2\frac{A_p^2}{A_e^2} - \frac{2(\rho_l g h)}{\rho_a}$$

Finally

$$V = \sqrt{2\frac{\rho_l}{\rho_a} g h \left(\frac{A_p^2}{A_e^2} - 1\right)^{-1}}$$

Using $\rho_l = 1000$ kg/m^3, $\rho_a = 1$ kg/m^3, $h = 0.1$ m, $A_p/A_e = (D_p/D_e)^2 = (5/0.3)^2$, we get $V = 0.159$ m/s.

Problem 2 (34 points)
Oil of density ρ and viscosity μ flows steadily in a rectangular channel (width into the page is much larger than height h) that is inclined at an angle α as shown in Figure. The flow is fully developed and the channel length is L. The channel length L is much larger than its height h so that the pressure can be approximated to be a function of x only. The inlet absolute pressure is $p_{in} > p_a$ and the outlet absolute pressure is atmospheric $p_{out} = p_a$.
(a) What is the equation governing the velocity? What are the boundary conditions?
(b) Find an expression of the velocity profile in terms of given quantities.
(c) What is the volume flow rate of oil?



Problem 2 Solution

(a) Conservation of momentum (steady state, fully developed, 2D):

$$\begin{aligned}
0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \rho g \sin\alpha \\
0 &= -\frac{\partial p}{\partial y} - \rho g \cos\alpha \\
0 &= -\frac{\partial p}{\partial z}
\end{aligned}$$

Resulting in $p(x, y) = -\rho g \cos\alpha\, y + f(x)$. Since $y \in [0, h]$ where $h << L$ we neglect the y dependence and assume that the pressure is a function of x only, i.e. $p(x)$. Integrating the first equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx} + \frac{g}{\nu}\sin\alpha \\
\Rightarrow\ &\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{dp}{dx} y + \frac{g}{\nu}\sin\alpha\, y + C_1 \\
\Rightarrow\ &u = \frac{1}{2\mu}\left(\frac{dp}{dx} + \rho g \sin\alpha\right) y^2 + C_1 y + C_2
\end{aligned}$$

The boundary conditions are : $u = 0$ at $y = 0$ and $u = 0$ at $y = h$ resulting in

$$\begin{aligned}
&C_2 = 0 \\
&\frac{1}{2\mu}\left(\frac{dp}{dx} + \rho g \sin\alpha\right) h^2 + C_1 h = 0 \Rightarrow C_1 = -\frac{1}{2\mu}\left(\frac{dp}{dx} + \rho g \sin\alpha\right) h
\end{aligned}$$

The velocity is then given by

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx} + \rho g \sin\alpha\right)(y^2 - hy)$$

The volume flow rate per unit width is

$$\begin{aligned}
\dot{\mathcal{V}} = \int_0^h u\, dy &= -\frac{1}{12\mu}\left(\frac{dp}{dx} + \rho g \sin\alpha\right) h^3 \\
&= -\frac{1}{12\mu}\left(\frac{p_{out} - p_{in}}{L} + \rho g \sin\alpha\right) h^3
\end{aligned}$$

Problem 3 (32 points)

When microorganisms or an aerosol particle(micron size solid or liquid particles) move through air or water, the Reynolds number is very small (Re<<1). In such low Re flows, the aerodynamic drag F_D on an object is a function of the mean diameter D_p of the object, fluid viscosity μ_a, gravity constant g, and density difference $\rho_p - \rho_a$, where ρ_p is the density of the particle and ρ_a is the density of air. Use dimensional analysis to express F_D in terms of the independent variables.

Problem 3 Solution

$F_D = f(D_p, \mu_a, g, \rho_p - \rho_a)$. The units of the various variables are:

$$\begin{aligned}
[F_D] &= \mathcal{M}\mathcal{L}\mathcal{T}^{-2} \\
[D_p] &= \mathcal{L} \\
[g] &= \mathcal{L}\mathcal{T}^{-2} \\
[\mu_a] &= \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1} \\
[\rho_p - \rho_a] &= \mathcal{M}\mathcal{L}^{-3}
\end{aligned}$$

We expect to have two dimensionless groups. Use D_p, g and μ as the repeating variables. The two groups are

$$\begin{aligned}
& \Pi_1 = F_D \, D_p^\alpha \, g^\beta \, \mu^\gamma \\
\Rightarrow \quad & 1 + \gamma = 0, 1 + \alpha + \beta - \gamma = 0, -2 - 2\beta - \gamma = 0 \\
\Rightarrow \quad & \gamma = -1, \beta = -\frac{1}{2}, \alpha = -\frac{3}{2} \\
\Rightarrow \quad & \Pi_1 = \frac{F_D}{\mu D_p^{3/2} g^{1/2}}
\end{aligned}$$

$$\Pi_2 = (\rho_p - \rho_a) \, D_p^\alpha \, g^\beta \, \mu^\gamma = \frac{(\rho_p - \rho_a) g^{1/2} D_p^{3/2}}{\mu}$$