Math 201 — Fall 2011–12

Calculus and Analytic Geometry III, all sections

Final Exam, January 21 — Duration: 2 hours 15 minutes

GRADES:

Problem	1 (/15)	2 (/15)	3 (/15)	4 (/11)	5 (
Part a					
Part b					
Part c					
Total					

NOTES BEFORE SOLVING THE EXAM:

1) You have to solve the recommended exercises in the book after understanding each chapter from the book if not from notes.

2) This exam is not a complete one and it needs around 90 minutes to be solved.

3) Please understand that this exam is solved by students, and it may contain some mistakes.

4) If you have any questions or concerns, email: insightclub@gmail.com

GRAND TOTAL: Solutions

GRADE:

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1
MWF 3, Kobeissi
Recitation F 11

Section 2
MWF 3, Kobeissi
Recitation F 5

Section 3
MWF 3, Kobeissi
Recitation F 4

Section 4
MWF 3, Kobeissi
Recitation F 10

Section 5
MWF 10, Abi-Khuzam
Recitation T 11

Section 6
MWF 10, Abi-Khuzam
Recitation T 3:30

Section 7
MWF 10, Abi-Khuzam
Recitation T 5

Section 8
MWF 10, Abi-Khuzam
Recitation T 2

Section 9
MWF 11, Brock
Recitation T 12:30

Section 10
MWF 11, Brock
Recitation T 2

Section 11
MWF 11, Brock
Recitation T 11

Section 12
MWF 11, Brock
Recitation T 3:30

Section 13
MWF 2, Nahlus
Recitation Th 11

Section 14
MWF 2, Nahlus
Recitation Th 3:30

Section 15
MWF 2, Nahlus
Recitation Th 8

Section 16
MWF 2, Nahlus
Recitation Th 5

Section 17
MWF 8, Makdisi
Recitation F 2

Section 18
MWF 8, Makdisi
Recitation Th 8

Section 19
MWF 8, Makdisi
Recitation Th 2

Section 20
MWF 8, Makdisi
Recitation Th 3:30

Section 21
MWF 1, Raji
Recitation M 8

Section 22
MWF 1, Raji
Recitation M 9

Section 23
MWF 1, Raji
Recitation M 4

Section 24
MWF 10, Egeileh
Recitation F 11

Section 25
MWF 10, Egeileh
Recitation F 2

Section 26
MWF 10, Egeileh
Recitation F 3

1. (5 pts each part, 15 pts total)

(a) Use the Sandwich theorem to find the following limit

$$l = \lim_{n\to\infty} \frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{3\ln\sqrt{n}}.$$

Let $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

$$\int_1^{n+1} f(x)\,dx < S_n < a_1 + \int_1^n f(x)\,dx$$

$$\int_1^{n+1} \frac{dx}{x} < S_n < 1 + \int_1^n \frac{dx}{x}$$

$$\ln(n+1) < S_n < 1 + \ln n$$

$$\frac{\ln(n+1)}{3\ln\sqrt{n}} < \frac{S_n}{3\ln\sqrt{n}} < \frac{1+\ln n}{3\ln\sqrt{n}} \to \frac{\ln(n+1)}{\frac{3}{2}\ln n} < \frac{S_n}{3\ln\sqrt{n}} < \frac{1+\ln(n)}{\frac{3}{2}\ln n}$$

as $n \to \infty$ the lower and upper bound $\to \frac{2}{3}$

$\therefore\ l = \frac{2}{3}$ by S. theorem

(b) (UNRELATED) Find, with justification, all values of p for which the following series is convergent

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin\left(\frac{1}{\sqrt{n}}\right)\right)^p.$$

$$\sin\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{\left(\frac{1}{\sqrt{n}}\right)^3}{3!} + \frac{\left(\frac{1}{\sqrt{n}}\right)^5}{5!} - \frac{\left(\frac{1}{\sqrt{n}}\right)^7}{7!} + \cdots$$

$$\left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right) = \frac{\left(\frac{1}{\sqrt{n}}\right)^3}{3!} - \frac{\left(\frac{1}{\sqrt{n}}\right)^5}{5!} + \frac{\left(\frac{1}{\sqrt{n}}\right)^7}{7!} - \cdots$$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)^p}{\left(\frac{1}{\sqrt{n}}\right)^{3p}} = \lim_{n\to\infty} \frac{\left(\left(\frac{1}{\sqrt{n}}\right)^3\frac{1}{3!} - \left(\frac{1}{\sqrt{n}}\right)^5\frac{1}{5!} + \left(\frac{1}{\sqrt{n}}\right)^7\frac{1}{7!} - \cdots\right)^p}{\left(\frac{1}{\sqrt{n}}\right)^{3p}}$$

$$= \frac{\cancel{\left(\frac{1}{\sqrt{n}}\right)^{3p}}\left(\frac{1}{3!} - \frac{\left(\frac{1}{\sqrt{n}}\right)^3}{5!} + \frac{\left(\frac{1}{\sqrt{n}}\right)^7}{7!} \cdots\right)^p}{\cancel{\left(\frac{1}{\sqrt{n}}\right)^{3p}}} = \frac{1}{6}$$

continued...

$\therefore$ by LCT the 2 series behave alike.

$\left(\frac{1}{\sqrt{n}}\right)^{3p}$ converges when $\frac{3}{2}p > 1$ (p-series)

$$p > \frac{2}{3}$$

So, does $\left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)^{p}$

(c) (UNRELATED) Compute the n^{th} partial sum S_n of the following series, and use it to find, according to the values of c, the sum of the series

$$S = \sum_{k=0}^{\infty} \frac{c^{k+1} - c^k}{(c^k + 3)(c^{k+1} + 3)}, \qquad c > 0.$$

a) $$S_n = \sum_{k=0}^{n} \frac{c^{k+1} - c^k}{(c^k+3)(c^{k+1}+3)} = \left(\frac{1}{4} - \cancel{\frac{1}{c+3}}\right) + \left(\cancel{\frac{1}{c+3}} - \cancel{\frac{1}{c^2+3}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{c^{n-1}+3}} - \cancel{\frac{1}{c^n+3}}\right) + \left(\cancel{\frac{1}{c^n+3}} - \frac{1}{c^{n+1}+3}\right)$$

$$= \frac{1}{4} - \frac{1}{c^{n+1}+3}$$

b) $$S = \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{c^{n+1}+3}\right) = \begin{cases} \text{if } |c|>1 & S = \frac{1}{4} \\ \text{if } |c|<1 & S = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12} \end{cases}$$

2. (5 pts each part, 15 pts total)

(a) Consider the function

$$f(x,y) = \frac{xy^2}{3\sin^2 x + y^2}, \text{ if } (x,y) \neq (0,0)$$

Prove that $f(0,0)$ may be defined in such a way that f becomes continuous at $(0,0)$.

$$0 \leq \sin^2 x \leq 1$$

$$y^2 \leq 3\sin^2 x + y^2 \leq 3 + y^2$$

$$\frac{1}{3\sin^2 x + y^2} \leq \frac{1}{y^2} \quad \text{multiply by } |xy^2|$$

$$\frac{|x|y^2}{3\sin^2 x + y^2} \leq |x|$$

by corrolary to sandwich thm, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ since $|x| \to 0$

(b) If in part (a) $f(0,0)$ has been defined correctly, prove that f is not differentiable at $(0,0)$. For this part, you may use without proof that $f_x(0,0) = f_y(0,0) = 0$.

continued...

(c) (UNRELATED) By about how much will $h(x,y) = \ln(x^2+y^2+z^2)$ change if the point $P(x,y,z)$ moves from $P_0(1,1,-1)$ a distance of $ds = 0.1$ unit in the direction of the vector $3\mathbf{i}+6\mathbf{j}-2\mathbf{k}$?

$$\nabla h = \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{K}$$

$$\nabla h|_{P_0} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{K}$$

$$\vec{u} = \frac{3i+6j-2k}{\|3i+6j-2k\|} = \frac{1}{\sqrt{3^2+6^2+(-2)^2}}(3\vec{i}+6\vec{j}-2\vec{K})$$

$$= \frac{1}{7}(3\vec{i}+6\vec{j}-2\vec{K})$$

$$= \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{K}$$

$$(\vec{\nabla} h|_{P_0}).\vec{u} = \frac{2}{7} + \frac{4}{7} + \frac{4}{21} = \frac{22}{21}$$

$$\Delta f = \frac{22}{21} \times ds = \frac{2.2}{21} \simeq 0.105$$

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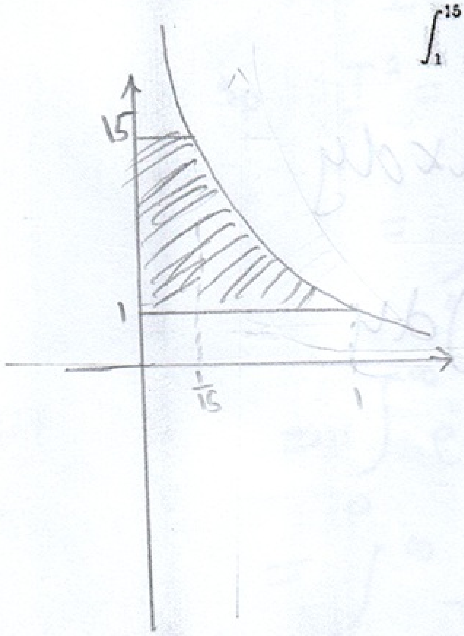
3. (5 pts each part, 15 pts total)

(a) Sketch the region of integration and evaluate the double integral

$$\int_1^{15}\int_0^{1/y} ye^{xy}\,dx\,dy. = \int_1^{15} e^{xy}\Big|_0^{1/y}\,dy$$

$$= \int_1^{15} e - e^0\,dy$$

$$= e-1\int_1^{15} dy = 14(e-1)$$

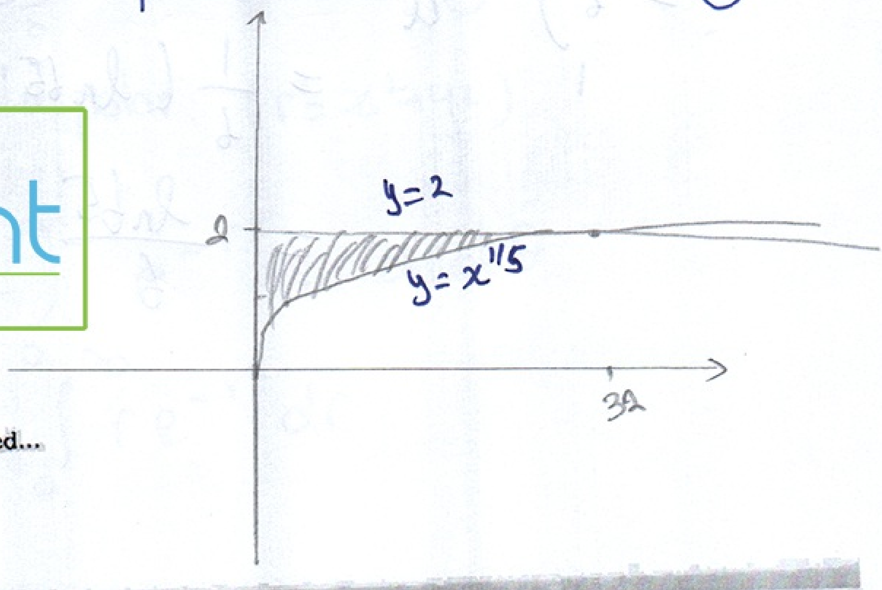


(b) Evaluate the double integral

$$\int_0^{32}\int_{x^{1/5}}^{2} \frac{dy\,dx}{y^6+1}.$$

we clearly don't know how to integrate this one.

So, we flip the order after sketching the region



continued...

$$I = \int_0^2 \int_0^{y^5} \frac{dx\,dy}{y^6+1}$$

$$= \int_0^2 \frac{1}{y^6+1} \int_0^{y^5} dx\,dy$$

$$= \int_0^2 \frac{1}{y^6+1} (y^5 - 0)\,dy$$

$$= \int_0^2 \frac{y^5}{y^6+1}\,dy$$

let $u = y^6 + 1$ $\quad \cdot\ u(0) = 1$

$du = 6y^5$ $\quad \cdot\ u(2) = 65$

$$= \frac{1}{6} \int_1^{65} \frac{du}{u} = \frac{1}{6} \Big[\ln|u| \Big]_1^{65}$$

$$= \frac{1}{6} (\ln 65 - \ln 1)$$

$$= \frac{\ln 65}{6}$$

(c) Evaluate the integral

$$I = \int_0^\infty e^{-x^2}\,dx.$$

so $$I^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-x^2}dx$$

$$= \int_0^\infty e^{-x^2}dx \cdot \underbrace{\int_0^\infty e^{-y^2}dy}_{\text{cst w.r.t } x\text{, call it } J}$$ (since x is a dummy variable)

$$= \int_0^\infty e^{-x^2}dx \times J$$

$$= \int_0^\infty J e^{-x^2}dx$$

$$= \int_0^\infty \int_0^\infty e^{-y^2}e^{-x^2}\,dy\,dx$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

Change to polar. ($r^2 = x^2+y^2$)

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r\,dr\,d\theta$$

$$= \frac{\pi}{2} \int_0^\infty r e^{-r^2}dr$$

$$= \frac{\pi}{2} \int_0^\infty r e^{-r^2} dr$$

let $u = +r^2$
$du = +2r\,dr$
$\left(\begin{array}{l} u(0) = 0 \\ u(\infty) = \infty \end{array} \right)$

$$= \frac{\pi}{2} \times \frac{1}{2} \int_0^\infty e^{-u} du$$

$$= \frac{\pi}{4} \left[-e^{-u} \right]_0^\infty$$

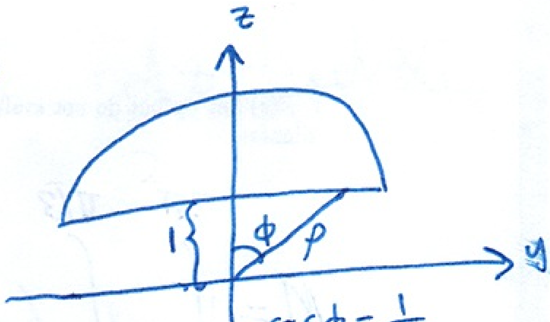
$$= \frac{\pi}{4} (-0 - (-1)) = \frac{\pi}{4} = I^2$$

So $I = \frac{\sqrt{\pi}}{2}$

4. (11 pts total: 5 pts for (a), 3 pts for (b), 3 pts for (c))

(a) Find the volume of the solid bounded below by the surface $z = 1$, and above by the surface $x^2 + y^2 + z^2 = 4$.

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{1/\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



$\cos\phi = \frac{1}{\rho}$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_{1/\cos\phi}^{2} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(\frac{8}{3} - \frac{1}{3\cos^3\phi}\right) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(\frac{8}{3}\sin\phi - \frac{\sin\phi}{3\cos^3\phi}\right) d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/3} \left(8\sin\phi - \frac{\tan\phi}{\cos^2\phi}\right) d\phi$$

$$\left(\begin{aligned} &u = \tan\phi \\ &du = \frac{d\phi}{\cos^2\phi} \\ &u(\phi = 0) = 0 \\ &u(\phi = \pi/3) = \sqrt{3} \end{aligned}\right)$$

$$= \frac{1}{3}\left[\theta\right]_0^{2\pi} \left[8\int_0^{\pi/3} \sin\phi \, d\phi - \int_0^{\sqrt{3}} u \, du\right]$$

$$= \frac{2\pi}{3}\left[-8\left[\cos\phi\right]_0^{\pi/3} - \left[\frac{u^2}{2}\right]_0^{\sqrt{3}}\right]$$

$$= \frac{2\pi}{3}\left[-8\left(-\frac{1}{2}\right) - \frac{3}{2}\right] = \frac{5\pi}{3}$$

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(b) If the density is $\delta(x,y,z) = z$, set up but do not evaluate, a triple integral with $dV = dx\,dy\,dz$, giving the mass of the solid in part (a).

$$M = \int_{1}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-\sqrt{4-y^2-z^2}}^{-\sqrt{4-y^2-z^2}} z \, dx \, dy \, dz$$

(c) Set up but do not evaluate the integral in part (b) in spherical coordinates.

$$M = \int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{1/\cos\phi}^{2} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

5. (14 pts total: 5 pts for (a), 5 pts for (b), 4 pts for (c))

(a) Let $f(x) = \frac{1}{x+3}$. Find the Taylor series expansion of f about $a = 1$, (i.e., centered at $a = 1$), and use it to find $f^{(n)}(1)$.

$$f(x) = \frac{1}{x+3} = \frac{1}{3+x-1+1} = \frac{1}{4+(x-1)}$$

$$= \frac{1}{4} \cdot \frac{1}{1+\frac{x-1}{4}} = \frac{1}{4}\sum_{0}^{\infty} \frac{(-(x-1))^n}{4^n}$$

$$= \frac{1}{4} \cdot \frac{1}{1-\left(\frac{-(x-1)}{4}\right)} \qquad = \sum_{0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x-1)^n$$

Now $f(x) = \sum_{0}^{\infty} c_n (x-1)^n$

$$\frac{f^{(n)}(1)}{n!} = c_n = \frac{(-1)^n}{4^{n+1}} \Rightarrow f^{(n)}(1) = \frac{(-1)^n n!}{4^{n+1}}$$

(b) (UNRELATED) Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 + x - y$ on the curve $x^2 + y^2 = 2$.

Use lagrange.

$g(x,y) = x^2+y^2-2$

$\nabla f = (2x+1, 2y-1)$

$\nabla g = (2x, 2y)$

$$\begin{cases} \nabla f = k\nabla g \\ x^2+y^2=2 \end{cases} \Rightarrow \begin{cases} 2x+1 = 2\lambda x & (1) \\ 2y-1 = 2\lambda y & (2) \\ x^2+y^2=2 & (3) \end{cases}$$

$(1) \Rightarrow x(2-2\lambda)+1=0 \Rightarrow x = \frac{1}{2\lambda-2}$

$(2) \Rightarrow y(2-2\lambda)+1=0 \Rightarrow y = \frac{1}{2\lambda-2}$

$\lambda \neq 1$

for $\lambda = 1$

$2x+1 = 2x$

$1 \neq 0$

continued...

$$\frac{1}{(2\lambda-2)^2}+\frac{1}{(2\lambda-2)^2}=2$$

$$\frac{2}{(2\lambda-2)^2}=2$$

$(2\lambda-2)^2=1$

$2\lambda-2=\pm 1$

$2\lambda=3$ or $2\lambda=1$

$\lambda=\frac{3}{2}$ or $\lambda=\frac{1}{2}$.

$\downarrow$ $\Downarrow$

$x=1$
$y=1$

$x=-1$
$y=-1$

$f(1,1)=2$

$f(-1,-1)=\cancel{0}\,2$

(c) (UNRELATED) Use the transformation $x = au, y = bv, z = cw$ to find the volume of the region $R = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$. Here a, b, c are positive constants.

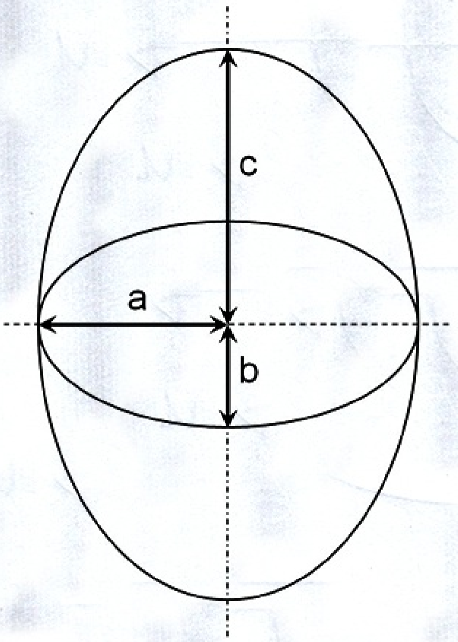
$x = au$
$y = bv$
$z = cw$

$$J(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= +a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0 \times \begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} + 0 \times \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix}$$

$$= abc$$



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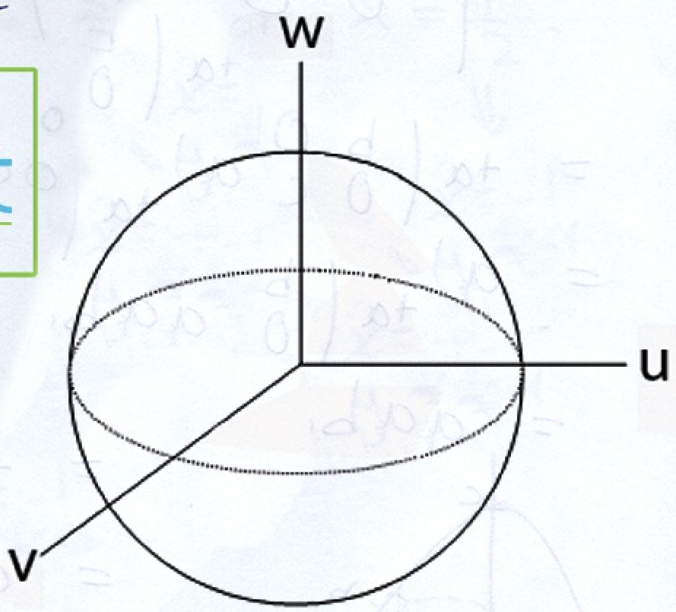
$$\text{Volume} = \iiint 1\, dV$$

$$= \iiint 1 . abc\ du\,dv\,dw$$

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

$$= \left\{ u^2 + v^2 + w^2 \leq 1 \right\}$$ = solid sphere of radius 1 in u-v-w plane.



$$\text{Volume} = abc \iiint_V 1\, du\,dv\,dw = abc \frac{4\pi}{3}(1)^3$$