EECE 442 Final Exam Solution Summer 2006

Problem # 1

Based on the spectrum of a PAM signal, the bandwidth of the signal is determined by that of the rectangular pulse whose duration is equal to T. Thus, the PAM signal bandwidth is equal to $\frac{1}{T}$ by the first zero crossing bandwidth definition. Hence, the smallest bandwidth is obtained when $T = T_s$. But, $\frac{1}{T_s} = 2 \times 5000 = 10{,}000$ Samples/sec

$\Rightarrow T = T_s = 10^{-4}$ sec $= 0.1$ ms.

Problem # 2

The pulse duration of a PDM signal is given by:

$$PD = K e^{[m(nT_s)/\max(m(nT_s))]}$$

Since the pulse duration in a PDM signal is proportional to the value of the sample from the message signal, then $(PD)_{max}$ is obtained when $m(nT_s) = \max(m(nT_s))$. Hence,

$$(PD)_{max} = T_s = K e^{[\max m(nT_s)/\max m(nT_s)]} = Ke$$

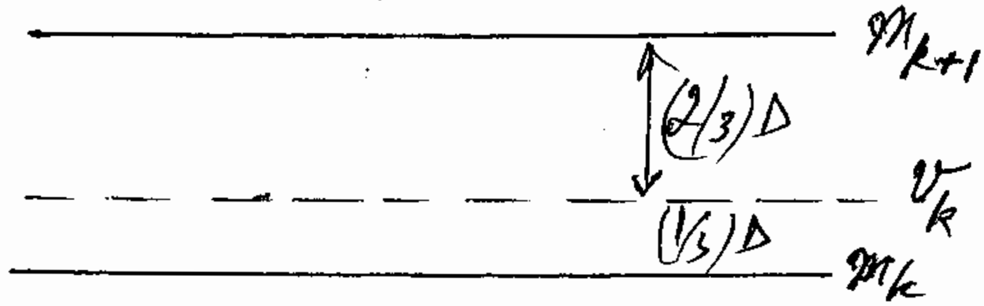
$$\Rightarrow K = T_s/e.$$

$$(PD)_{min} = \frac{T_s}{e} e^{[\min m(nT_s)/\max m(nT_s)] X(0)}$$

$$= \frac{T_s}{e} e^{[-2/2]} = \frac{T_s}{e^2}.$$

Here, we used $\min(m(nT_s)) = -2$ and $\max(m(nT_s)) = 2$ since the message has a dynamic range between -2 volts and 2 volts and since the sampling rate is much bigger than the Nyquist rate. This means that T_s is very small and sample values at -2 and 2 volts (or at very close values) will be obtained.

Problem # 3



Based on the above representation of v_k in cell (m_k, m_{k+1}), the range of the quantization noise, represented by $Q = M - V$ is $\left[-\frac{1}{3}\Delta, \frac{2}{3}\Delta\right]$.

Now, since Q is uniformly distributed, then

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{1}{3}\Delta \le q \le \frac{2}{3}\Delta \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(Q^2) = \int_{-1/3\,\Delta}^{2/3\,\Delta} \frac{1}{\Delta} q^2 \, dq = \frac{1}{3\Delta} q^3 \Big|_{-\frac{1}{3}\Delta}^{\frac{2}{3}\Delta}$$

$$= \frac{1}{3\Delta}\left[\left(\frac{2}{3}\right)^3 \Delta^3 + \left(\frac{1}{3}\right)^3 \Delta^3\right]$$

$$= \frac{\Delta^2}{3}\left[\frac{8+1}{27}\right] = \frac{\Delta^2}{9}$$

Problem #4

Based on the result in problem #3; i.e., $E(Q^2) = \frac{\Delta^2}{9}$, and the fact that $E[Q^2] = \frac{\Delta^2}{12}$ when v_k is the midpoint of cell (m_k, m_{k+1}), then it is better to place v_k at the midpoint of the cell because it gives a smaller average power of the quantization noise. The midpoint is also better than all other ~~indicated~~ positions of v_k in cell (m_k, m_{k+1}) since this can be checked easily by calculating $E(Q^2)$.

Problem # 5

The compression of the analog signal reduces its dynamic range and this allows the use of a smaller number of quantization levels compared to the number of levels used for the uncompressed signal.

Problem # 6

$$\text{The bit rate in PCM} = \frac{\log_2 L}{(T_s)_{PCM}} = \frac{\log_2 256}{(T_s)_{PCM}}$$

$$= 8 \times 2 \times 5000 = 80{,}000 \text{ bits/sec}$$

$$\text{The bit rate in DM} = \frac{1}{(T_s)_{DM}} = 80{,}000 \text{ bits/sec.}$$

$\Rightarrow$ Sampling rate in DM $= 80{,}000$ Samples/sec.

Problem # 7

$$\text{Bit rate in PCM} = \frac{\log_2\left(\frac{16}{0.0625}\right)}{T_s}$$

$$= \frac{\log_2 256}{T_s} = \frac{8}{T_s}.$$

$$\text{Bit rate in DPCM} = \frac{3}{4} \times \frac{8}{T_s} = \frac{\log_2\left(\frac{DR}{0.0625}\right)}{T_s}$$

Hence, $\log_2\left(\frac{DR}{0.0625}\right) = 6$

$\Rightarrow DR = 0.0625 \times 2^6 = 4$ Volts.

Problem #8

$$(SNR)_{PCM} = \frac{\sigma_M^2}{\frac{(\Delta)^2_{PCM}}{12}} = \frac{12\,\sigma_M^2}{\left(\frac{16}{256}\right)^2}$$

$$(SNR)_{DPCM} = \frac{\sigma_M^2}{\frac{(\Delta)^2_{DPCM}}{12}} = \frac{12\,\sigma_M^2}{\left(\frac{8}{256}\right)^2}$$

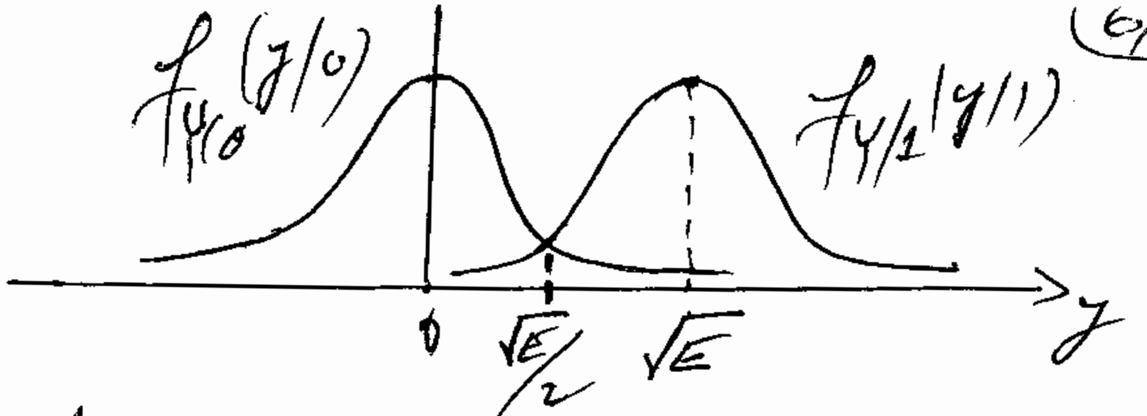
$$\Rightarrow (SNR)_{DPCM} = 4\,(SNR)_{PCM}$$

Problem #9

$$Y = \begin{cases} \sqrt{E} + W, & \text{if bit 1 is sent} \\ W, & \text{if bit 0 is sent} \end{cases}$$

$$f_{Y|1}(y/1) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(y-\sqrt{E})^2}{N_0/2}\right\}$$

$$f_{Y|0}(y/0) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{y^2}{N_0/2}\right\}.$$



Under $p_0 = p_1 = 1/2$, the minimum probability of error criterion gives the maximum likelihood decision rule. Hence, $\lambda = \frac{\sqrt{E}}{2}$.

Problem #10

$$y(iT_b) = \sum_{k=1}^{i} A_k \, p[(i-k)T_b]$$

$$y(3T_b) = \underbrace{A_1 p[2T_b] + A_2 p(T_b)}_{\text{ISI term.}} + \underbrace{A_3 p(0)}_{\text{Signal term}}$$

$A_3 = -A$ since third bit is 0

$A_2 = A$ since 2nd bit is 1

$A_1 = -A$ since first bit is 0.

With $p(t) = e^{-|t|}$, for all t, then:

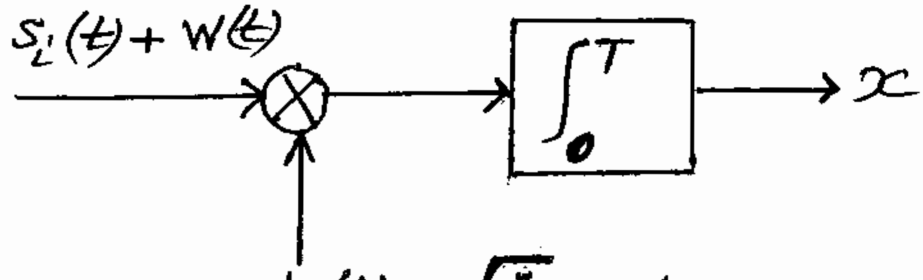
$$\text{ISI term} = -Ae^{-2T_b} + Ae^{-T_b}$$

$$= A\left[e^{-T_b} - e^{-2T_b}\right].$$

Problem # 11

$$s_i(t) = i\sqrt{\frac{2E}{T}}\cos(\omega_c t), \quad 0 \le t \le T,$$

where $i = -2, -1, 1, 2$.



$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(\omega_c t)$$

$$x = i\int_0^T \frac{2\sqrt{E}}{T}\cos^2(\omega_c t)\,dt + \int_0^T W(t)\phi_1(t)\,dt$$

$$= i\frac{\sqrt{E}}{T}\left[\int_0^T dt + \int_0^T \cos(2\omega_c t)\,dt\right] + W_1$$

$$= i\sqrt{E} + W_1, \quad i = -2, -1, 1, 2.$$

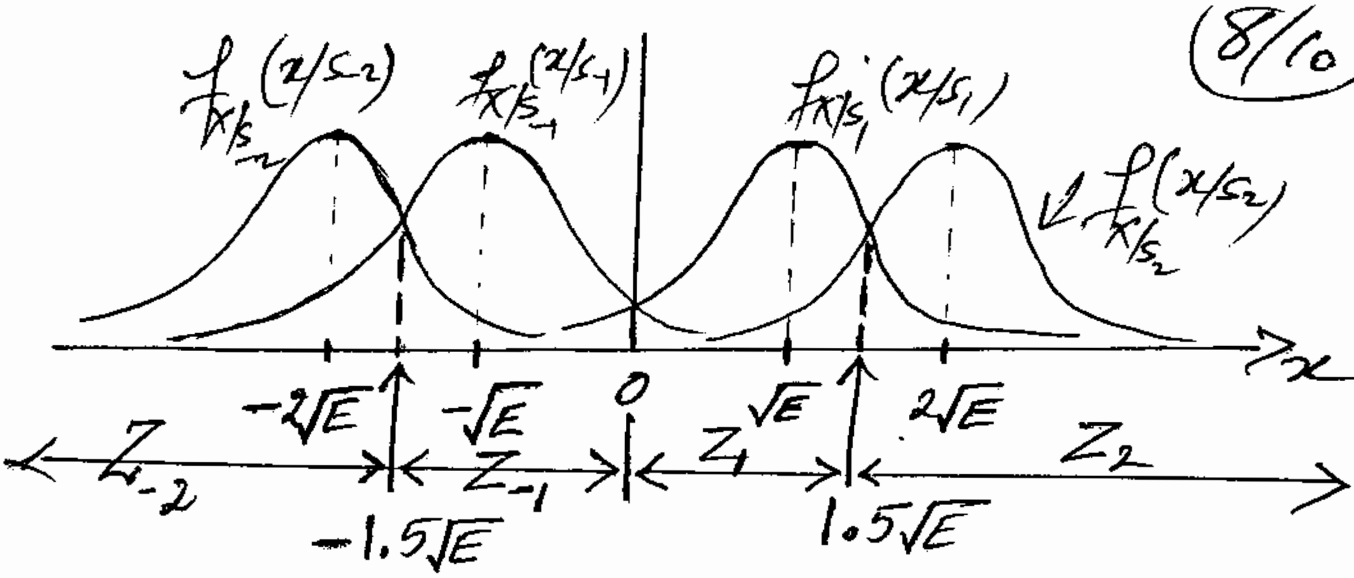
W_1 is a zero mean Gaussian random variable with variance equal to $N_0/2$. Hence, the probability density functions of the random variable X under the transmission of $s_i(t)$, $i = -2, -1, 1, 2$ are:

$$f_{X/S_{-2}}(x/s_{-2}) = \frac{1}{\sqrt{\pi N_0}}\exp\left\{-\frac{1}{2}\frac{(x + 2\sqrt{E})^2}{N_0/2}\right\}$$

$$f_{X/S_{-1}}(x/S_{-1}) = \frac{1}{\sqrt{\pi N_0}}\exp\left\{-\frac{1}{2}\frac{(x + \sqrt{E})^2}{N_0/2}\right\}$$

$$f_{X/S_1}(x/s_1) = \frac{1}{\sqrt{\pi N_0}}\exp\left\{-\frac{1}{2}\frac{(x - \sqrt{E})^2}{N_0/2}\right\}$$

$$f_{X/S_2}(x/S_2) = \frac{1}{\sqrt{\pi N_0}}\exp\left\{-\frac{1}{2}\frac{(x - 2\sqrt{E})^2}{N_0/2}\right\}$$



The minimum probability of error criterion under equally probable transmitted signal gives the maximum likelihood decision rule.

Hence, the decision rule becomes:

If $0 < x < 1.5\sqrt{E}$, decide $s_1(t)$.

$-1.5\sqrt{E} < x < 0$, decide $s_{-1}(t)$

$1.5\sqrt{E} < x < \infty$, decide $s_2(t)$

$-\infty < x < -1.5\sqrt{E}$, decide $s_{-2}(t)$.

Of course, the above decision rule can also be obtained using the general structure of decision making device as obtained in class notes.

Problem # 12

$s_1(t) = \sqrt{\frac{2E}{T}} \cos(\omega_c t), \quad 0 \le t \le T$

$s_2(t) = \sqrt{\frac{2E}{T}} \cos(\omega_c t - \frac{\pi}{2}), \quad 0 \le t \le T$

$s_3(t) = \sqrt{\frac{2E}{T}} \cos(\omega_c t + \frac{\pi}{2}), \quad 0 \le t \le T$

$s_4(t) = \sqrt{\frac{2E}{T}} \cos(\omega_c t + \pi), \quad 0 \le t \le T$

$s_4(t)$ is linearly dependent on $s_1(t)$.

$s_3(t)$ is linearly dependent on $s_2(t)$.

Hence, according to the Gram-Schmidt procedure, $s_1(t)$ and $s_2(t)$ can be used to construct 2 orthonormal functions $\phi_1(t)$ and $\phi_2(t)$.

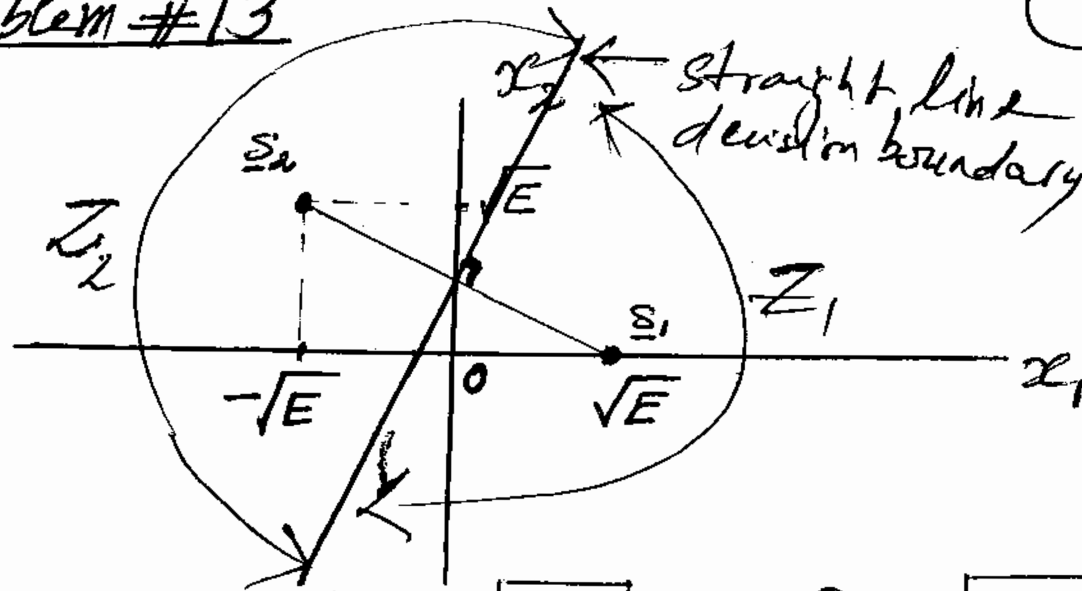
$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t), \quad 0 \le t \le T$

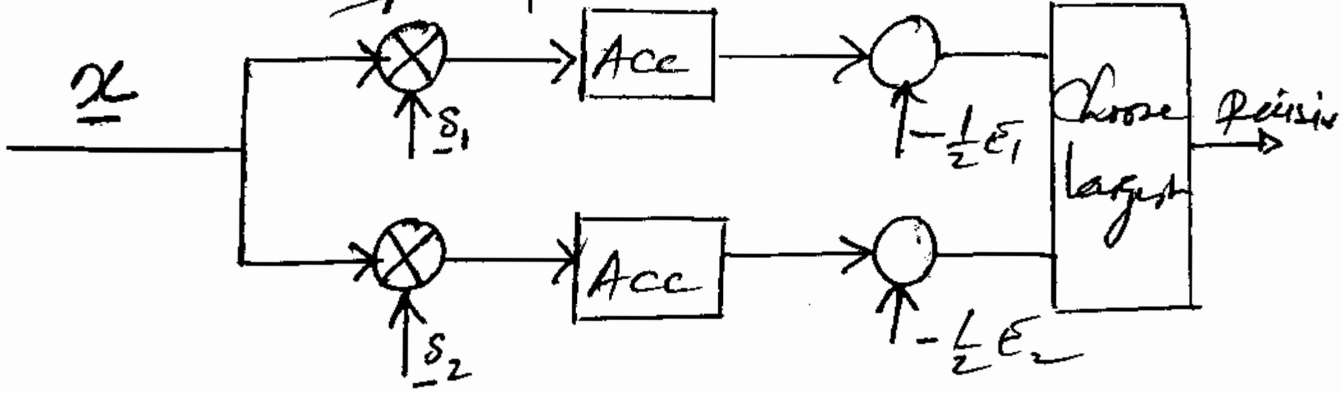
$\phi_2(t) = \sqrt{\frac{2}{T}} \cos(\omega_c t - \pi/2),$

$\quad = \sqrt{\frac{2}{T}} \sin(\omega_c t), \quad 0 \le t \le T.$

Hence, 2 correlators need to be used in the reception of the given 4 signals in additive white and Gaussian noise.

Problem #13





$$\underline{x} \cdot \underline{s}_1 - \frac{1}{2}E_1 = \sqrt{E}\,x_1 - \frac{1}{2}E \quad (1)$$

$$\underline{x} \cdot \underline{s}_2 - \frac{1}{2}E_2 = -\sqrt{E}\,x_1 + \sqrt{E}\,x_2 - \frac{1}{2}(2E) \quad (2)$$

The straight line between the decision regions is obtained by equating (1) and (2).

$$\Rightarrow 2\sqrt{E}\,x_1 + \frac{1}{2}E = \sqrt{E}\,x_2$$

$$\Rightarrow x_2 = 2x_1 + \frac{1}{2}\sqrt{E}.$$