Problem 3.31

Let the sinusoidal signal $m(t) = A\sin\omega_0 t$, where $\omega_0 = 2\pi f_0$

The autocorrelation of the signal is

$$R_m(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

$$R_m(0) = \frac{A^2}{2}$$

$$R_m(1) = \frac{A^2}{2}\cos\left(\omega_0 \times \frac{1}{10 f_0}\right)$$

$$= \frac{A^2}{2}\cos\left(\frac{\pi}{5}\right)$$

For this problem, we thus have

$$\mathbf{R}_m = [R_m(0)], \quad \mathbf{r}_m = [R_m(1)]$$

(a) The optimum solution is given by

$$\mathbf{w}_0 = \mathbf{R}_m^{-1}\mathbf{r}_m$$

$$= \frac{\frac{A^2}{2}\cos(\pi/5)}{\frac{A^2}{2}} = \cos(\pi/5)$$

$$= ~~0.995~~\ 0.809$$

(b) $J_{\min} = R_m(0) - \mathbf{r}_m^T\mathbf{R}_m^{-1}\mathbf{r}_m$

$$= \frac{A^2}{2} - \frac{A^2}{2}\cos(\pi/5) \times \frac{A^2}{2}\cos(\pi/5)/(A^2/2)$$

2/5

$$= \frac{A^2}{2}(1 - \cos^2(\not{0.1}))$$

$\frac{\pi}{5}$

$$= \not{0.005}\,A^2$$

$0.345\,A^2$

Problem 3.32

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}$$

$$\mathbf{r}_x = [0.8,\ 0.6,\ 0.4]^T$$

(a) $\mathbf{w}_0 = \mathbf{R}_x^{-1}\mathbf{r}_x$

$$= \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix}$$

(b) $J_{\min} = R_x(0) - \mathbf{r}_x^T\mathbf{R}_x^{-1}\mathbf{r}_x$

$$= R_x(0) - \mathbf{r}_x^T\mathbf{w}_0$$

$$= 1 - [0.8,\ 0.6,\ 0.4] \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix}$$

$$= 1 - (0.8 \times 0.875 - \ 0.4 \times \ 0.125)$$

$$= 1 - 0.7 + 0.05$$

$$= 0.35$$

Problem 3.33

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

$$\mathbf{r}_x = [0.8, \quad 0.6]^T$$

(a) $\mathbf{w}_0 = \mathbf{R}_x^{-1}\mathbf{r}_x$

$$= \begin{bmatrix} 0.8889 \\ -0.1111 \end{bmatrix}$$

(b) $J_{\min} = R_x(0) - \mathbf{r}_x^T\mathbf{R}_x^{-1}\mathbf{r}_x$

$$= 1 - 0.6444$$

$$= 0.3556$$

which is slightly worse than the result obtained with a linear predictor using three unit delays (i.e., three coefficients). This result is intuitively satisfying.

Problem 3.34

Input signal variance = $R_x(0)$

The normalized autocorrelation of the input signal for a lag of one sample interval is

$$\rho_x(1) = \frac{R_x(1)}{R_x(0)} = 0.75$$

Error variance $= R_x(0) - R_x(1)R_x^{-1}(0)R_x(1)$

$$= R_x(0)(1 - \rho_x^2(1))$$

$$\text{Processing gain} = \frac{R_x(0)}{R_x(0)(1-\rho_x^2(1))}$$

$$= \frac{1}{1-\rho_x^2(1)}$$

$$= \frac{1}{1-(0.75)^2}$$

$$= 2.2857$$

Expressing the processing gain in dB, we have

$$10\log_{10}(2.2857) = 3.59 \text{ dB}$$

Problem 3.35

$$\text{Processing gain} = \frac{R_x(0)}{R_x(0)\left(1-\frac{\mathbf{r}_x^T\mathbf{R}_x^{-1}\mathbf{r}_x}{R_x(0)}\right)}$$

(a) Three-tap predictor:
Processing gain = 2.8571
= 4.56 dB

(b) Two-tap predictor:
Processing gain = 2.8715
= 4.49 dB

Therefore, the use of a three-tap predictor in the DPCM system results an improvement of 4.56 - 4.49 = 0.07 dB over the corresponding system using a two-tap predictor.

Problem 3.36

(a) For DPCM, we have $10\log_{10}(\text{SNR})_0 = \alpha + 6n$ dB

For PCM, we have $10\log_{10}(\text{SNR})_0 = 4.77 + 6n - 20\log_{10}(\log(1+\mu))$

where n is the number of quantization levels
SNR of DPCM

SNR = $\alpha + 6n$, where $-3 < \alpha < 15$

For n=8, the SNR is in the range of 45 to 63 dBs.

SNR of PCM

$$\begin{aligned} SNR &= 4.77 + 6n - 20\log_{10}(\log(2.56)) \\ &= 4.77 + 48 - 14.8783 \\ &= 38 \text{ dB} \end{aligned}$$

Therefore, the SNR improvement resulting from the use of DPCM is in the range of 7 to 25 dB.

(b) Let us assume that n_1 bits/sample are used for DPCM and n bits/sample for PCM

If $\alpha = 15$ dB, then we have

$$15 + 6n_1 = 6n - 10.0$$

Rearranging: $$(n - n_1) = \frac{10 + 15}{6}$$

$$= 4.18$$

which, in effect, represents a saving of about 4 bits/sample due to the use of DPCM.

If, on the other hand, we choose $\alpha = -3$ dB, we have

$$-3 + 6n_1 = 6n - 10$$

Rearranging: $$(n - n_1) = \frac{10 - 3}{6}$$

$$= \frac{7}{6}$$

$$= 1.01$$

which represents a saving of about 1 bit/sample due to the use of DPCM.