CHAPTER 4

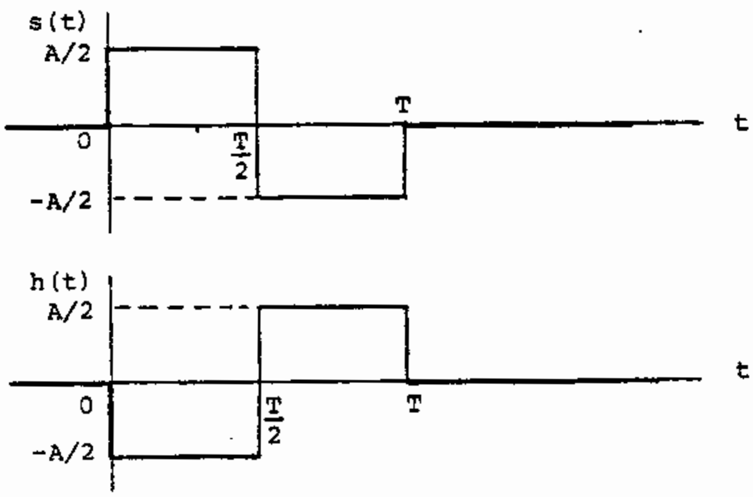
Baseband Pulse Transmission

Problem 4.1

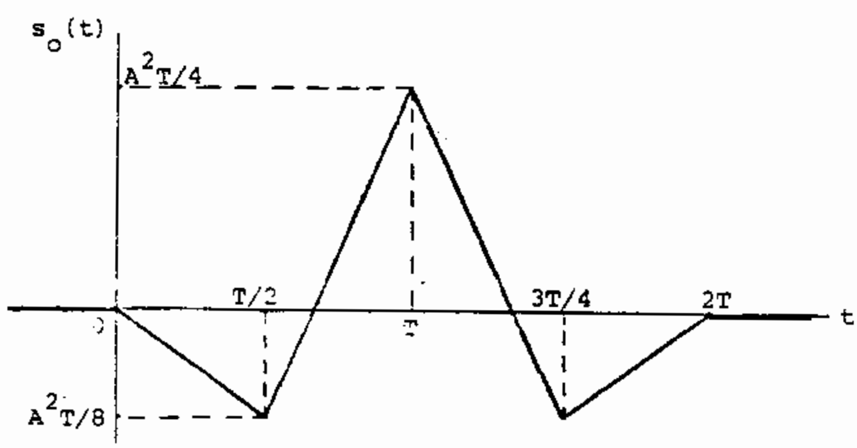
(a) The impulse response of the matched filter is

$h(t) = s(T-t)$

The $s(t)$ and $h(t)$ are shown below:



(b) The corresponding output of the matched filter is obtained by convolving $h(t)$ with $s(t)$. The result is shown below:



The peak value of the filter output is equal to $A^2T/4$, occuring at $t=T$.

The typewritten pages are retrieved from the Solution Manual of Haykin's Book: Comm. Systems, 4th Edition, Wil

Problem 4.2

(a) The matched filter of impulse response $h_1(t)$ for pulse $s_1(t)$ is given in the solution to Problem 4.1. The matched filter of impulse response $h_2(t)$ for $s_2(t)$ is given by

$$h_2 = s_2(T-t)$$

which has the following waveform:

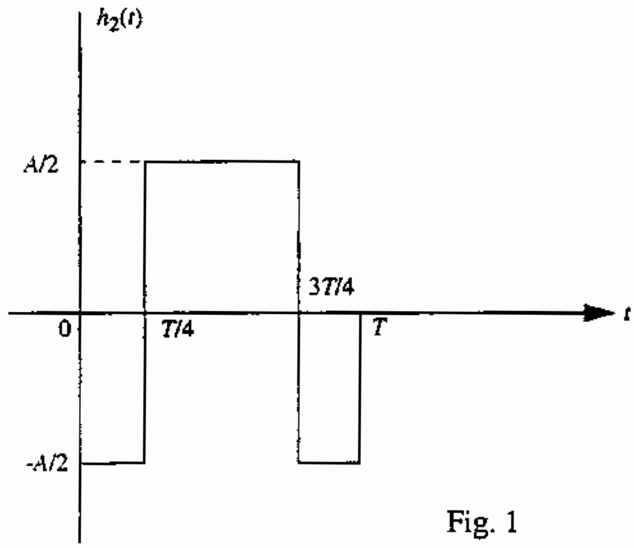


Fig. 1

(b) (i) The response of the matched filter, matched to $s_2(t)$ and due to $s_1(t)$ as input, is obtained by convolving $h_2(t)$ with $s_1(t)$, as shown by

$$y_{21}(t) = \int_0^T s_1(\tau)h_2(t-\tau)d\tau$$

The waveform of the output $y_{21}(t)$ so computed is plotted in Figure 2. This figure also includes the corresponding waveforms of input $s_1(t)$ and impulse response $h_2(t)$.

(ii) Next, the response of the matched filter, matched to $s_1(t)$ and due to $s_2(t)$ as input, is obtained by convolving $h_1(t)$ with $s_2(t)$, as shown by

$$y_{12}(t) = \int_0^T s_2(\tau)h_1(t-\tau)d\tau$$

Figure 3 shows the waveforms of input $s_2(t)$, impulse response $h_1(t)$, and response $y_{12}(t)$.

4.2(b)
(i)

Pulse $s_1(t)$

Filter $h_2(t)$

$h_2(t) = s_2(T-t)$

Filter response

$\int_0^T s_1(\tau)\, h_2(t-\tau)\, d\tau$

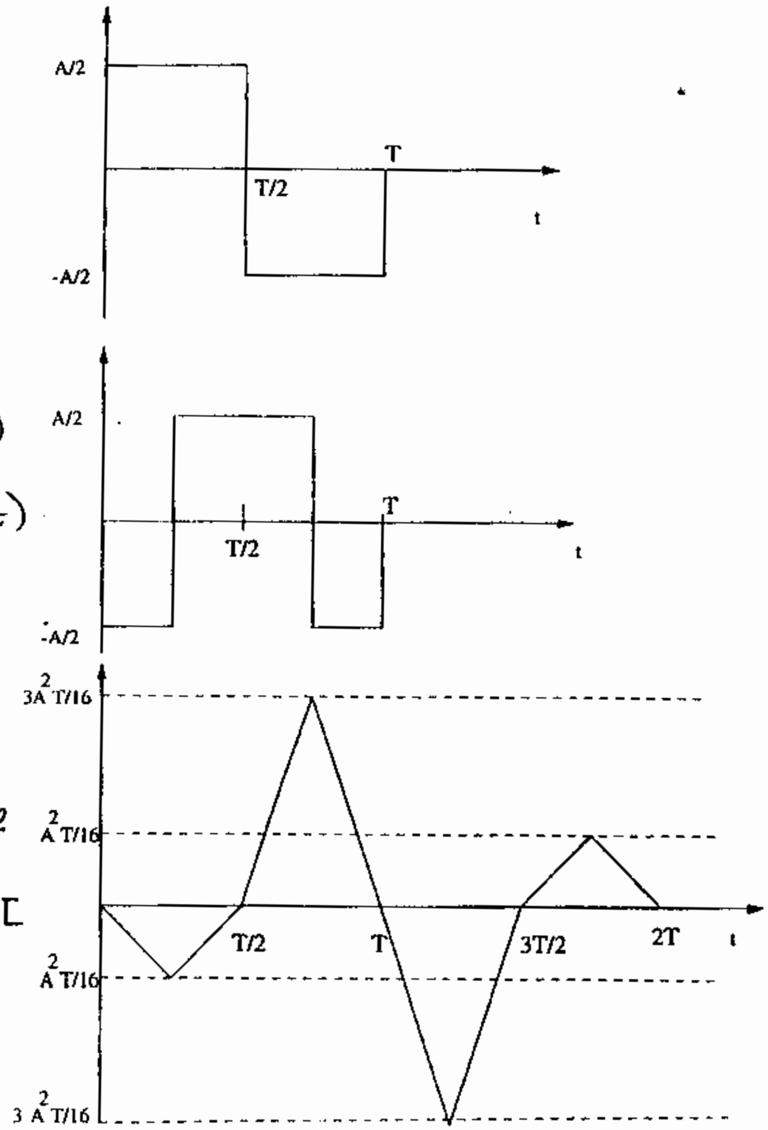


Fig. 3

4.2(b)

(ii)

Pulse $s_2(t)$

Filter $h_1(t)$

Filter matched to Pulse $s_1(t)$

Filter response

$\int_0^T s_2(\tau) h_1(t-\tau) d\tau$

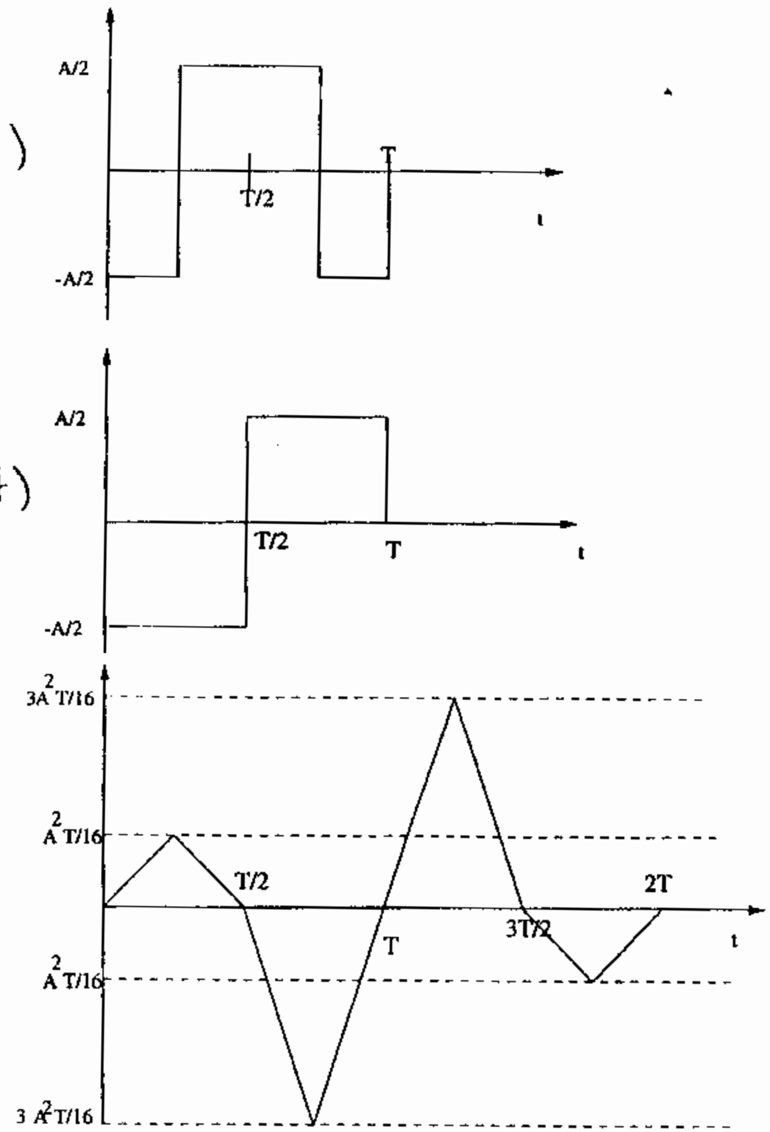


Fig. 3

Note that $y_{12}(t)$ is exactly the negative of $y_{21}(t)$. However, in both cases we find that at $t = T$, both outputs are equal to zero, as shown by

$$y_{21}(T) = y_{12}(T) = 0$$

For n pulses $s_1(t)$ $s_2(t)$,...,$s_n(t)$ that are orthogonal to each other over the interval [0,T], the n-dimensional matched filter has the following structure:

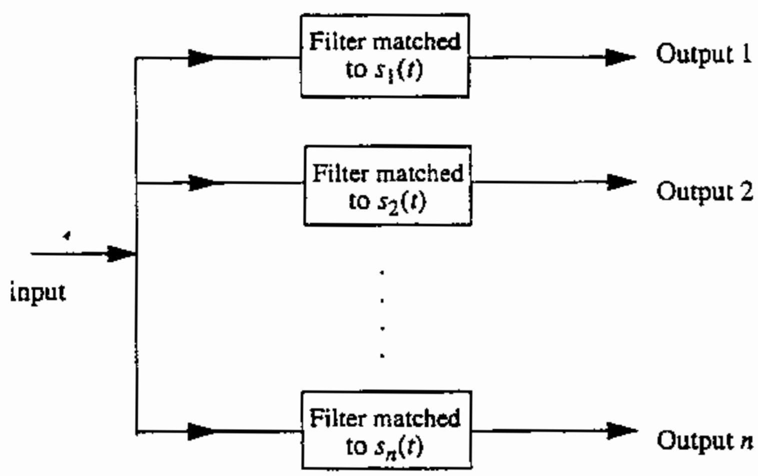
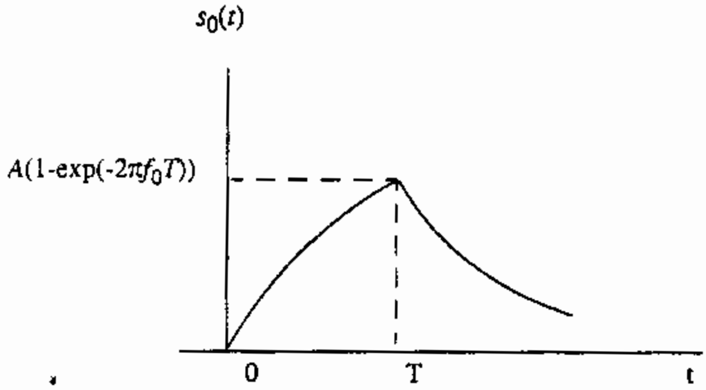


Fig. 4

Problem 4.4

The output of the low-pass RC filter, produced by a rectangular pulse of amplitude A and duration T, is as shown below:



The peak value of the output pulse power is

$$P_{\text{out}} = A^2[1 - \exp(-2\pi f_0 T)]^2$$

where f_0 is the 3-dB cutoff frequency of the RC filter.

The average output noise power is

$$N_{\text{out}} = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{df}{1 + (f/f_0)^2}$$

$$= \frac{N_0 \pi f_0}{2}$$

The corresponding value of the output signal-to-noise ratio is therefore

$$(\text{SNR})_{\text{out}} = \frac{2A^2}{N_0 \pi f_0}[1 - \exp(-2\pi f_0 T)]$$

Differentiating $(\text{SNR})_0$ with respect to $f_0 T$ and setting the result equal to zero, we find that $(\text{SNR})_{\text{out}}$ attains its maximum value at

$$f_0 = \frac{0.2}{T}$$

The corresponding maximum value of $(\text{SNR})_{\text{out}}$ is

$$(\mathrm{SNR})_{0,\max} = \frac{2A^2T}{0.2\pi N_0}[1 - \exp(-0.4\pi)]^2$$

$$= \frac{1.62A^2T}{N_0}$$

For a perfect matched filter, the output signal-to-noise ratio is

$$(\mathrm{SNR})_{0,\mathrm{matched}} = \frac{2E}{N_0}$$

$$= \frac{2A^2T}{N_0}$$

Hence, we find that the transmitted energy must be increased by the ratio 2/1.62, that is, by 0.92 dB so that the low-pass RC filter with $f_0 = 0.2/T$ realizes the same performance as a perfectly matched filter.

Problem 4.6

The average probability of error is

$$P_e = p_1 \int_{-\infty}^{\lambda} f_Y(y|1)dx + p_0 \int_{\lambda}^{\infty} f_Y(y|0)dx \tag{1}$$

An optimum choice of λ corresponds to minimum P_e. Differentiating Eq. (1) with respect to λ, we get:

$$\frac{\partial P_e}{\partial \lambda} = p_1 f_Y(\lambda|1) - p_0 f_Y(\lambda|0)$$

Setting $\frac{\partial P_e}{\partial \lambda} = 0$, we get the following condition for the optimum value of λ:

$$\frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda)_{opt}|0)} = \frac{p_0}{p_1}$$

which is the desired result.

Problem 4.7

In a binary PCM system, with NRZ signaling, the average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The signal energy per bit is

$$E_b = A^2 T_b$$

where A is the pulse amplitude and T_b is the bit (pulse) duration. If the signaling rate is doubled, the bit duration T_b is reduced by half. Correspondingly, E_b is reduced by half.

Let $u = \sqrt{E_b/N_0}$. We may then set

$$P_e = 10^{-6} = \frac{1}{2}\,\text{erfc}(u)$$

Solving for u, we get

$$u = 3.3$$

When the signaling rate is doubled, the new value of P_e is

$$P_e' = \frac{1}{2}\,\text{erfc}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\,\text{erfc}(2.33)$$

$$= 10^{-3}.$$

Problem 4.8

(a) The average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = A^2T_b$. We may rewrite this formula as

$$P_e = \frac{1}{2}\,\text{erfc}\left(\frac{A}{\sigma}\right) \qquad (1)$$

where A is the pulse amplitude at $\sigma = \sqrt{N_0T_b}$. We may view σ^2 as playing the role of noise variance at the decision device input. Let

$$u = \sqrt{\frac{E_b}{N_0}} = \frac{A}{\sigma}$$

We are given that

$$\sigma^2 = 10^{-2}\ \text{volts}^2, \qquad \sigma = 0.1\ \text{volt}$$

$$P_e = 10^{-8}$$

Since P_e is quite small, we may approximate it as follows:

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

We may thus rewrite Eq. (1) as (with $P_e = 10^{-8}$)

$$\frac{\exp(-u^2)}{2}\sqrt{\pi}u = 10^{-8}$$

Solving this equation for u, we get

$$u = 3.97$$

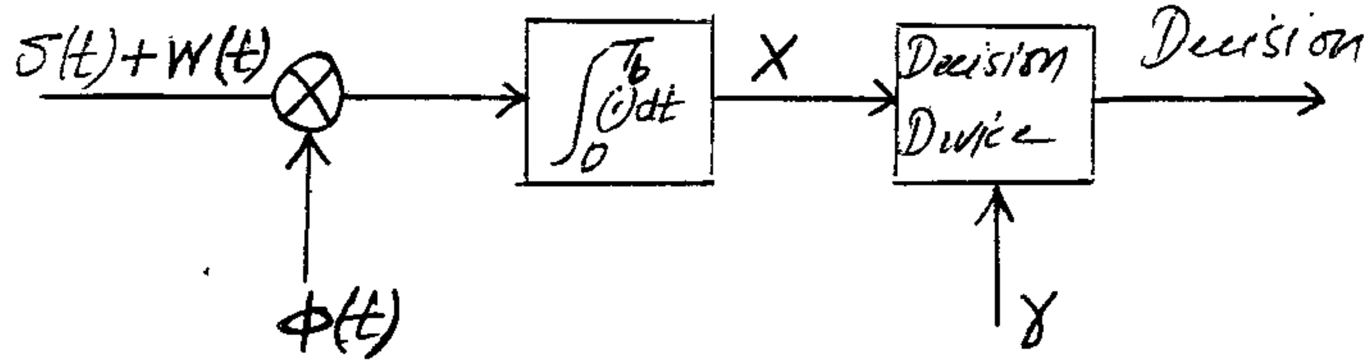
The corresponding value of the pulse amplitude is

$$A = \sigma u = 0.1 \times 3.97$$

$$= 0.397 \text{volts}$$

Problem 4.9

For the unipolar NRZ signaling using a rectangular pulse of amplitude A and duration T_b to represent bit 1 the receiver structure is as follows:



$$s(t) = \begin{cases} A, & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \text{ if bit 1 is transmitted}$$

$$s(t) = 0, \quad 0 \le t \le T_b \text{ if bit 0 is transmitted.}$$

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T_b}}, & 0 \le t \le T_b \\ 0, & \text{elsewhere} \end{cases} \text{ Normalized rectangular pulse.}$$

$$\Rightarrow X = \begin{cases} \int_0^{T_b} A \times \frac{1}{\sqrt{T_b}} dt + \int_0^{T_b} \frac{1}{\sqrt{T_b}} w(t)\, dt, & \text{if 1 is sent} \\ \int_0^{T_b} \frac{1}{\sqrt{T_b}} w(t)\, dt & \text{if 0 is sent.} \end{cases}$$

$$= \begin{cases} A\sqrt{T_b} + W & \text{if 1 is sent} \\ W & \text{if 0 is sent.} \end{cases}$$

$$X = \begin{cases} \sqrt{E_b} + W & \text{if 1 is sent} \\ W & \text{if 0 is sent} \end{cases}$$

With $E_b = A^2 T_b$ being the energy of the rectangular pulse representing 1, and $W = \frac{1}{\sqrt{T_b}} \int_0^{T_b} w(t)\,dt$

With $W(t)$ being a zero mean white and Gaussian noise process with $N_0/2$ as its spectral height, then, W is a zero mean Gaussian random variable.

$$\sigma_W^2 = E(W^2) = E\left[\frac{1}{T_b}\int_0^{T_b} w(t)\,dt \int_0^{T_b} w(u)\,du\right]$$

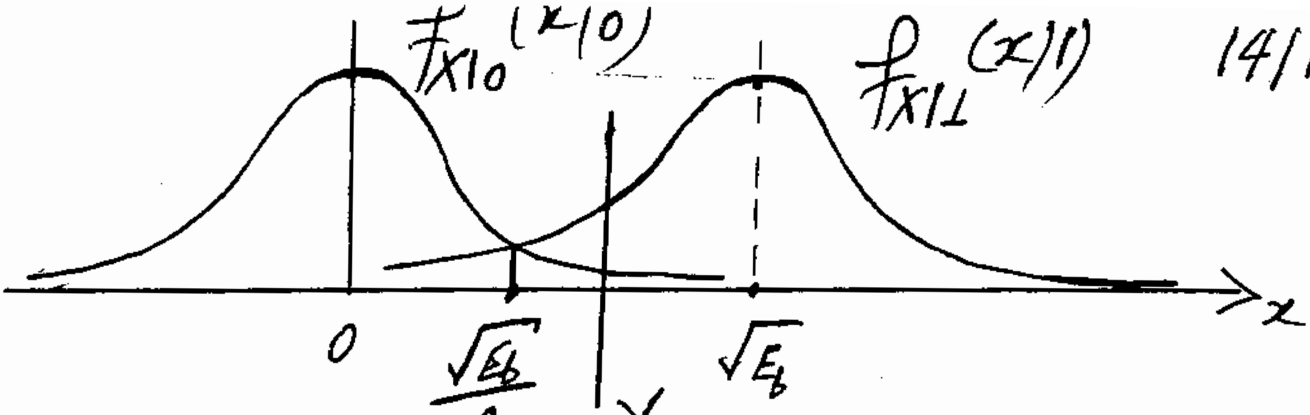
$$= \frac{1}{T_b}\int_0^{T_b} dt \int_0^{T_b} E\left[w(t)w(u)\right] du$$

$$= \frac{1}{T_b}\int_0^{T_b} dt \int_0^{T_b} \frac{N_0}{2}\,\delta(t-u)\,du$$

$$= \frac{1}{T_b}\int_0^{T_b} \frac{N_0}{2}\,dt = \frac{N_0}{2}$$

Hence, $f_{X|1}(x/1) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(x-\sqrt{E_b})^2}{N_0/2}\right\}$

$$f_{X|0}(x/0) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{x^2}{N_0/2}\right\}$$



$$p_e = \frac{1}{2}\int_{-\infty}^{\gamma} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(x-\sqrt{E_b})^2}{N_0/2}\right\} dx$$

$$+ \frac{1}{2}\int_{\gamma}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(x^2)}{N_0/2}\right\} dx$$

$$\frac{dP_e}{d\gamma} = \frac{1}{2}\times\frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(\gamma-\sqrt{E_b})^2}{N_0/2}\right\}$$

$$-\frac{1}{2}\times\frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{\gamma^2}{N_0/2}\right\} = 0$$

Upon cancellation of common terms and applying the natural logarithm, the following can be obtained:

$$(\gamma-\sqrt{E_b})^2 = \gamma^2 \Rightarrow \gamma = \frac{\sqrt{E_b}}{2}$$

Replacing γ in the above expression of p_e we get:

$$P_{e,min} = \int_{-\infty}^{\frac{\sqrt{E_b}}{2}} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(x-\sqrt{E_b})^2}{N_0/2}\right\} dx$$

Let: $\dfrac{x-\sqrt{E_b}}{\sqrt{N_0/2}} = y \Rightarrow$

$$P_{e,min} = \int_{-\infty}^{-\frac{\sqrt{E_b}}{2}/\sqrt{N_0/2}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y^2\right\} dy$$

$$\Rightarrow P_{e,min} = \mathrm{erf}_*\left(-\sqrt{\frac{E_b}{2N_0}}\right)$$

With: $\mathrm{erfc}(x) = 2\,\mathrm{erf}_*(-\sqrt{2}\,x)$,

$$P_{e,min} = \frac{1}{2}\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

Problem 4.10

For unipolar RZ signaling, we have

Binary symbol 1: $s(t) = +A$ for $0 < t \leq T_b/2$
and $s(t) = 0$ for $T_b/2 < t \leq T_b$

Binary symbol 0: $s(t) = 0$ for $0 < t \leq T$

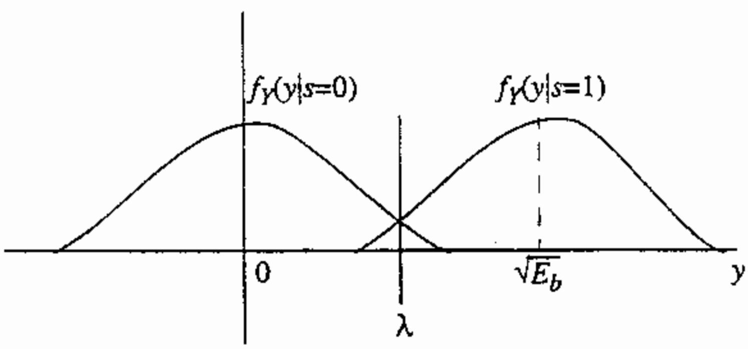
The a priori probabilities of symbols 1 and 0 are assumed to be equal, in which case we have $p_0 = p_1 = 1/2$.

To determine the average probability of error, we consider the two possible kinds of error separately. We begin by considering the first kind of error that occurs when symbol 0 is sent and the receiver chooses symbol 1. In this case, the probability of error is just the probability that the matched filter output will exceed the threshold λ owing to the presence of noise, so the transmitted symbol 0 is mistaken for symbol 1.

$$\text{Energy of symbol 1} = \frac{A^2 T_b}{2} = E_b$$

Energy of symbol 0 = 0

The conditional probability density function of the two signals is given below:



With symbols 1 and 0 assumed to be equiprobable, the optimum threshold is

$$\lambda = \frac{1}{2}\sqrt{E_b} = \frac{1}{2}\sqrt{\frac{A^2 T_b}{2}}$$

Given that symbol 0 was transmitted, the probability of error is simply the probability that $y > \lambda$, as shown by

$$P(\text{error}|0) = \int_{\lambda}^{\infty} f_Y(y|0)dy$$

$$= \frac{1}{\sqrt{\pi N_0}}\int_{\lambda}^{\infty} \exp\left(-\frac{y^2}{N_0}\right)dy$$

Define a new variable z as

$$z = \frac{y}{\sqrt{N_0}}$$

We then have

$$P(\text{error}|0) = \frac{1}{\sqrt{\pi}}\int_{\lambda/\sqrt{N_0}}^{\infty} \exp(-z^2)dz$$

$$= \frac{1}{2}\text{erfc}\left(\frac{\lambda}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2T_b}{2N_0}}\right)$$

Similarly, $P(\text{error}|1) = \int_{-\infty}^{\lambda} f_Y(y|(1)dy$

$$= \frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{\lambda} \exp\left[-\frac{(y-\sqrt{E_b})^2}{N_0}\right]dy$$

Define $z = \frac{\sqrt{E_b}-y}{\sqrt{N_0}}$, and so write

$$P(\text{error}|1) = \frac{1}{\sqrt{\pi}}\int_{\frac{\sqrt{E_b}-\lambda}{\sqrt{N_0}}}^{\infty} \exp(-z^2)dz$$

$$P(\text{error}|1) = \frac{1}{2}\text{erfc}\left(\frac{\sqrt{E_b} - \lambda}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{\sqrt{E_b}}{2\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{2N_0}}\right)$$

The average probability of error is therefore

$$P_e = P(1)P(\text{error}|1) + P(\text{error}|0)P(0)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{E_b/N_0}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{2N_0}}\right) \qquad (1)$$

The average probability of error for on-off (i.e., unipolar NRZ) type of encoded signals is

$$\frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{N_0}}\right)$$

Comparing this result with that of Eq. (1) for the unipolar RZ type of encoded signals, we immediately see that, for a prescribed noise spectral density N_0, the symbol energy in unipolar RZ signaling has to be doubled in order to achieve the same average probability of error as in unipolar NRZ signaling.