EECE442 Homework #7 Solution

Problem 5.1

(a) Unipolar NRZ code.

The pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively are defined by

$$s_1(t) = \sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b \quad ; \quad E_b = A^2 T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

where E_b is the transmitted signal energy per bit and T_b is the bit duration. From the definitions of $s_1(t)$ and $s_2(t)$, it is clear that, in the case of unipolar NRZ signals, there is only one basis function of unit energy. The basis function is given by

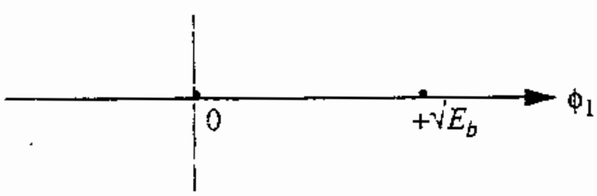
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, we may expand the transmitted signals $s_1(t)$ and $s_2(t)$ in terms of $\phi_1(t)$ as follows:

$$s_1(t) = \sqrt{E_b}\,\phi_1(t), \qquad 0 \le t \le T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for unipolar NRZ code is $(+\sqrt{E_b}, 0)$, as shown



(b) Polar NRZ code.

In this code, binary symbols 1 and 0 are defined by

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$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b \quad ; \quad E_b = A^2 T_b$$

$$s_2(t) = -\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

The basis function is given by

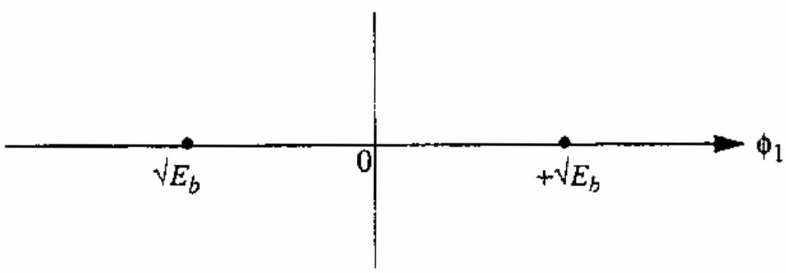
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, the transmitted signals in terms of $\phi_1(t)$ are as follows:

$$s_1(t) = \sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

$$s_2(t) = -\sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for the polar NRZ code is $(+\sqrt{E_b}, -\sqrt{E_b})$ as shown below:



(c) Unipolar return-to-zero code.

In this third code, binary symbols 1 and 0 are defined by

$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b/2 \quad ; \quad E_b = A^2 T_b$$

$$= 0 \qquad T_b/2 \le t \le T_b$$

$$s_2(t) = 0 \qquad 0 \le t \le T_b$$

The energy of signal $s_1(t)$ is

$$E_1 = \int_0^{T_b/2} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt + \int_{T_b/2}^{T_b} 0 dt$$

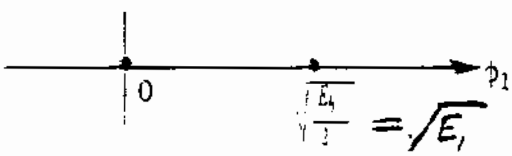
$$= \frac{E_b}{2} = A^2 T_b / 2$$

The energy of signal $s_2(t)$ is zero.

The basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\frac{E_b}{2}}}$$

The signal-space diagram for the RZ code is as follows:



Problem 5.2

The given 8-level PAM signal is defined by

$$s_i(t) = A_i \text{rect}\left(\frac{t}{T} - \frac{T}{2}\right)$$

The energy of signal $s_i(t)$ is given by

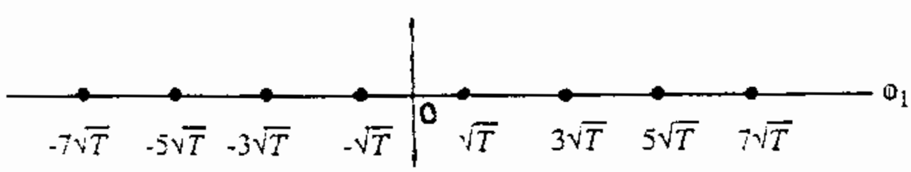
$$E_i = \int_0^T (A_i)^2 dt$$

$$= A_i^2 T, \qquad A_i = \pm 1, \pm 3, \pm 5, \pm 7$$

The basis function is given by

$$\phi_i(t) = \frac{s_i(t)}{\sqrt{E_i}} = \frac{s_i(t)}{A_i \sqrt{T}}$$

The signal-space diagram of the 8-level PAM signal is as follows:



Problem 5.4

(a) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

$$E_1 = \int_0^1 (2)^2 dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwsie} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$= \int_0^1 (-4)(1)dt = -4$$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$= \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}$$

$$= \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{31} = \int_0^T s_3(t)\ \phi_1(t)dt$$

$$= \int_0^1 (3)(1)dt = 3$$

$$s_{32} = \int_T^{2T} s_3(t)\ \phi_2(t)dt$$

$$= \int_1^2 (3)(-1)dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\ \phi_1(t) - s_{32}\ \phi_2(t)$$

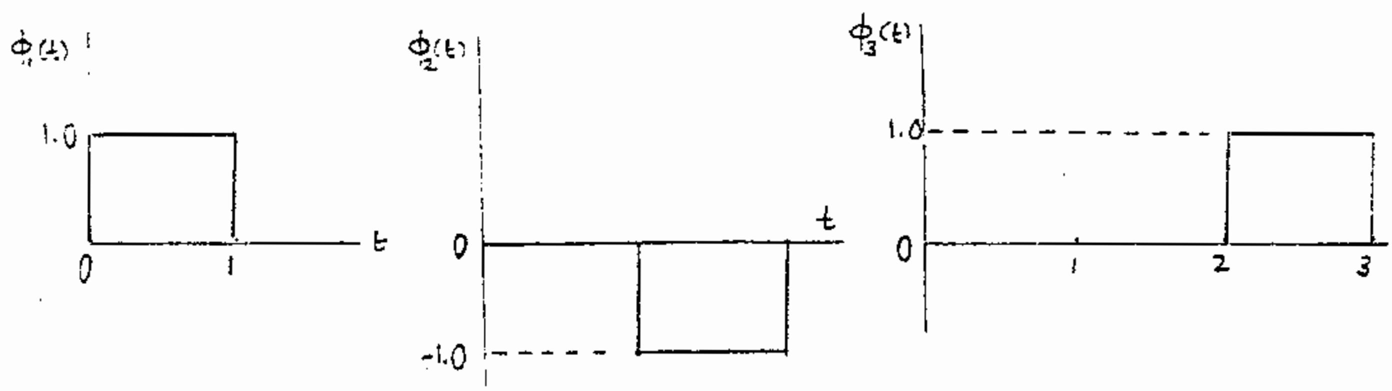
$$= \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the thrid basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

The three basis functions are as follows (graphically)



(b) $s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

Problem 5.5

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T$$

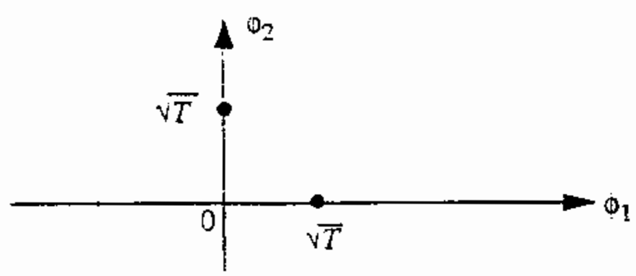
To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown below:



Problem 5.7

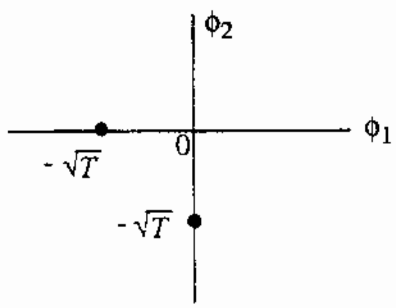
(a) The biorthogonal signals are defined as the negatives of orthogonal signals. Consider for example the two orthogonal signals $s_1(t)$ and $s_2(t)$ defined as follows:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

$$s_2(t) = \sqrt{E}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal basis functions. The biorthogonal signals are given by $-s_1(t)$ and $-s_2(t)$, which are respectively expressed in terms of the basis functions as $-\sqrt{E}\phi_1(t)$ and $-\sqrt{E}\phi_2(t)$. Hence, the inclusion of these two biorthogonal signals leaves the dimensionality of the signal-space diagram unchanged. This result holds for the general case of M orthogonal signals.

(b) The signal-space diagram for the biorthogonal signals corresponding to those shown in Fig. P5.5 is as shown in Fig. 1a. Incorporating this diagram with that of the solution to Problem 5.5, we get the 4-signal constellation shown in Fig. 1b.



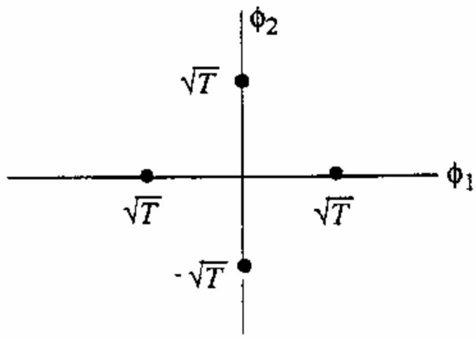


Figure 1

Problem 5.8

(a) A pair of signals $s_i(t)$ and $s_k(t)$, belonging to an N-dimensional signal space, can be represented as linear combinations of N orthonormal basis functions. We thus write

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{matrix} 0 \le t \le T \\ i = 1, 2 \end{matrix} \tag{1}$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array} \tag{2}$$

The real-valued basis functions $\phi_1(t)$ and $\phi_2(t)$ are orthonormal. Hence,

$$\int_0^T \phi_i(t)\phi_j(t) \quad = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional vector defined by

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots, M \tag{4}$$

where M is the number of signals in the setl, with $M \geq N$. The inner product of the pair of signal $s_i(t)$ and $s_k(t)$ is given by

$$\int_0^T s_i(t)s_k(t)dt \tag{5}$$

By substituting (1) in (5), we get the following result for the inner product:

$$\int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right] \left[\sum_{l=1}^{N} s_{kl}\phi_l(t)\right]dt$$

$$= \sum_{j=1}^{N} \sum_{l=1}^{N} s_{ij}s_{kl}\int_0^T \phi_j(t)\phi_l(t)dt \tag{6}$$

Since the $\phi_j(t)$ form an orthonormal set, then, in accordance with the two conditions of Eq. (3) and (4), the inner product of $s_i(t)$ and $s_k(t)$ reduces to

$$\int_0^T s_i(t)s_k(t)dt = \sum_{j=1}^{N} s_{ij}s_{kj}$$

$$= \mathbf{s}_i^T \mathbf{s}_k$$

(b) Consider next the squared Euclidean distance between $\mathbf{s}_i$ and $\mathbf{s}_k$, which can be expressed as follows:

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = (\mathbf{s}_i - \mathbf{s}_k)^T (\mathbf{s}_i - \mathbf{s}_k)$$

$$= \mathbf{s}_i^T \mathbf{s}_i + \mathbf{s}_k^T \mathbf{s}_k - 2\mathbf{s}_i^T \mathbf{s}_k$$

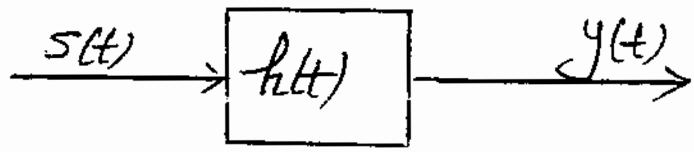
$$= \int_0^T s_i^2(t)dt + \int_0^T s_k^2(t)dt - 2\int_0^T s_i(t)s_k dt$$

$$= \int_0^T (s_i(t) - s_k(t))^2 dt$$

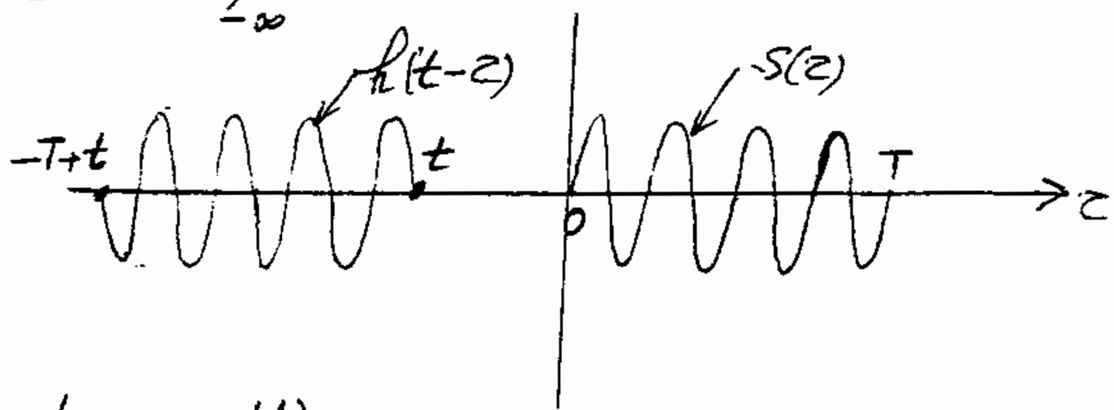
Problem 5.11

a) $$y(T)=\int_0^T s^2(t)\,dt=\int_0^T \sin^2\left(\frac{8\pi}{T}t\right)dt$$
$$=\frac{1}{2}\int_0^T\left(1-\cos\left(\frac{16\pi}{T}t\right)\right)dt=\frac{T}{2}$$

b)



$$h(t)=s(T-t)$$
$$=\sin\left[\frac{8\pi}{T}(T-t)\right]=-\sin\frac{8\pi t}{T},\quad 0\le t\le T.$$
$$y(t)=\int_{-\infty}^{\infty}s(\tau)h(t-\tau)\,d\tau$$



$t\le 0,\ y(t)=0$

$$0\le t<T,\ y(t)=\int_0^t s(\tau)h(t-\tau)\,d\tau=-\int_0^t \sin\left(\frac{8\pi}{T}\tau\right)\sin\left(\frac{8\pi}{T}(t-\tau)\right)d\tau$$
$$\Rightarrow y(t)=-\frac{1}{2}\left[\int_0^t\cos\left(\frac{8\pi}{T}(2\tau-t)\right)d\tau-\int_0^t\cos\left(\frac{8\pi}{T}t\right)d\tau\right]$$
$$=-\frac{1}{2}\left[\frac{T}{16\pi}\sin\left(\frac{8\pi}{T}(2\tau-t)\right)\Big|_0^t-t\cos\left(\frac{8\pi}{T}t\right)\right]$$
$$=-\frac{1}{2}\left[\frac{T}{16\pi}\left(\sin\left(\frac{8\pi}{T}t\right)+\sin\left(\frac{8\pi}{T}t\right)\right)-t\cos\left(\frac{8\pi}{T}t\right)\right]$$
$$=\frac{-T}{16\pi}\sin\left(\frac{8\pi}{T}t\right)+\frac{1}{2}t\cos\left(\frac{8\pi}{T}t\right)$$

$T \le t \le 2T$, $y(t) = \int_{-T+t}^{T} s(\tau) h(t-\tau) d\tau = -\int_{-T+t}^{T} \sin\left(\frac{8\pi}{T}\tau\right) \sin\left(\frac{8\pi}{T}(t-\tau)\right) d\tau$

$$\Rightarrow y(t) = -\frac{1}{2}\left[\frac{T}{16\pi} \sin\left(\frac{8\pi}{T}(2\tau - t)\right)\Big|_{-T+t}^{T} - \tau \cos\left(\frac{8\pi t}{T}\right)\Big|_{-T+t}^{T}\right]$$

$$= -\frac{1}{2}\left[\frac{T}{16\pi} \times 2 \sin\left(\frac{8\pi}{T}t\right) - (2T - t) \cos\frac{8\pi t}{T}\right]$$

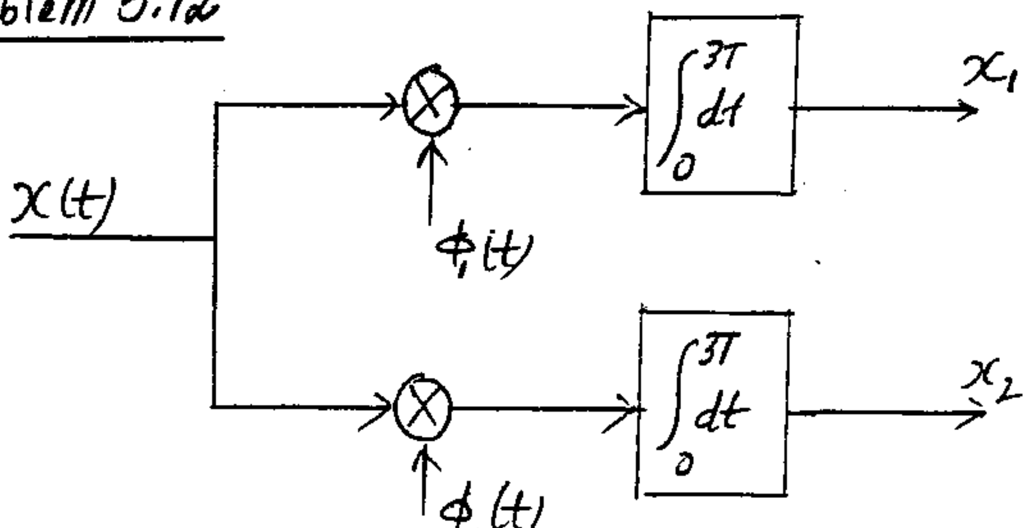
$$= \frac{T}{16\pi} \sin\left(\frac{8\pi}{T}t\right) + \left(T - \frac{t}{2}\right) \cos\left(\frac{8\pi}{T}t\right)$$

$t \geq 2T$, $y(t) = 0$

c) At time $t = T$, $y(t) = \frac{T}{2}$ which is equal to the correlator output.

Problem 5.12

a)



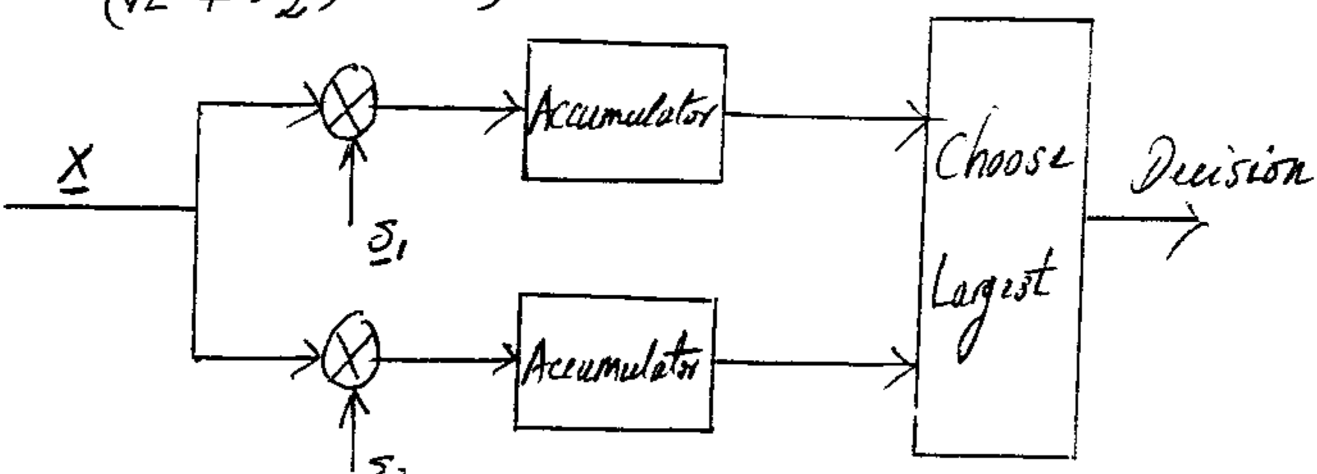
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}\ ;\ E_1 = \int_0^{3T} s_1^2(t)\,dt = \int_0^{3T} dt = 3T$$

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}}\ ;\ E_2 = \int_0^{3T} s_2^2(t)\,dt = 3T = E_1 = E$$

$$x_1 = \int_0^{3T} x(t)\phi_1(t)\,dt = \begin{cases} \int_0^{3T} s_1(t)\phi_1(t)\,dt + \int_0^{3T} w(t)\phi_1(t)\,dt, & k=1 \\ \int_0^{3T} s_2(t)\phi_1(t)\,dt + \int_0^{3T} w(t)\phi_1(t)\,dt, & k=2 \end{cases}$$

$$= \begin{Bmatrix} \sqrt{E} + w_1, & k=1 \\ 0 + w_1, & k=2 \end{Bmatrix},\quad w_1 = \int_0^{3T} w(t)\phi_1(t)\,dt$$

$$x_2 = \begin{Bmatrix} 0 + w_2, & k=1 \\ \sqrt{E} + w_2, & k=2 \end{Bmatrix},\quad w_2 = \int_0^{3T} w(t)\phi_2(t)\,dt$$



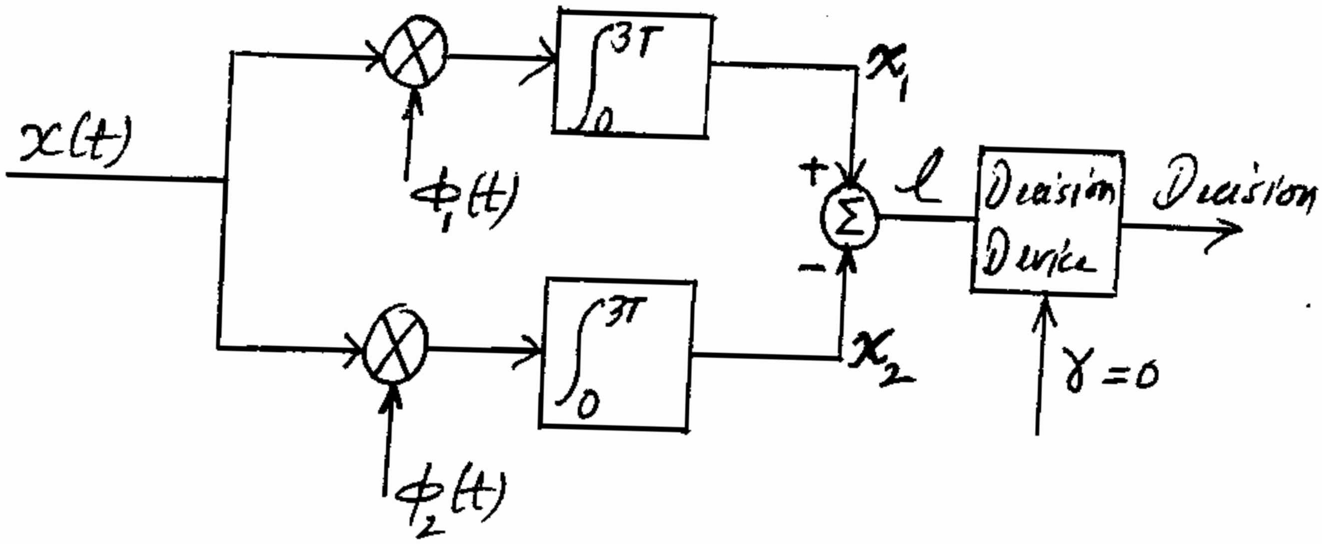
$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \underline{s}_1 = \begin{bmatrix} \sqrt{E} \\ 0 \end{bmatrix}, \quad \underline{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E} \end{bmatrix}$$

$$\left.\begin{array}{l} \underline{X}^T \cdot \underline{s}_1 = \sqrt{E}\, X_1 \\ \underline{X}^T \cdot \underline{s}_2 = \sqrt{E}\, X_2 \end{array}\right\} \Rightarrow$$

The decision is as follows:

If $X_1 - X_2 > 0$, decide $s_1(t)$

If $X_1 - X_2 < 0$, decide $s_2(t)$.



b) $l = x_1 - x_2$.

$$L = X_1 - X_2 = \begin{cases} \sqrt{E} + (W_1 - W_2), & k = 1 \\ -\sqrt{E} + (W_1 - W_2), & k = 2 \end{cases}$$

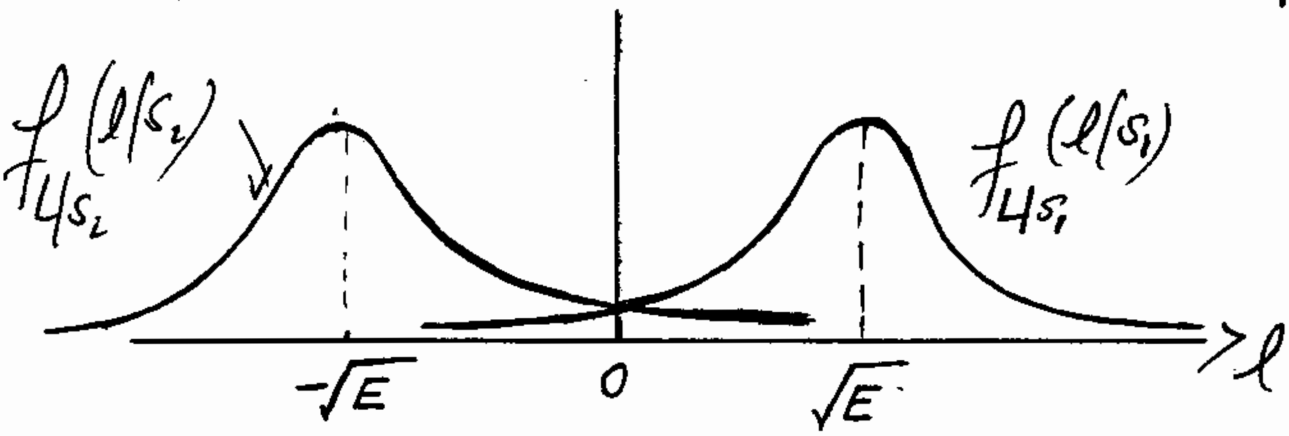
W_1 is a zero mean Gaussian r.v. with variance $N_0/2$,

W_2 " " " " " " " $N_0/2$,

$W = W_1 - W_2$ is also a zero mean Gaussian random variable.

$$\sigma_W^2 = E\left[(W_1 - W_2)^2\right] = E[W_1^2] + E[W_2^2] - \underbrace{2E(W_1 W_2)}_{=0}.$$

$$= \frac{N_0}{2} + \frac{N_0}{2} = N_0$$



$$f_{L/s_1}(l/s_1) = \frac{1}{\sqrt{2\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(l-\sqrt{E})^2}{N_0}\right\}$$

$$f_{L/s_2}(l/s_2) = \frac{1}{\sqrt{2\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(l+\sqrt{E})^2}{N_0}\right\}$$

$$p_e = \frac{1}{2}\int_{-\infty}^{0} f_{L/s_1}(l/s_1)\,dl + \frac{1}{2}\int_{0}^{\infty} f_{L/s_2}(l/s_2)\,dl$$

$$= \int_{-\infty}^{0} f_{L/s_1}(l/s_1)\,dl = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi N_0}} \exp\left\{-\frac{1}{2}\frac{(l-\sqrt{E})^2}{N_0}\right\} dl$$

Let: $\frac{l-\sqrt{E}}{\sqrt{N_0}} = y \Rightarrow$

$$p_e = \int_{-\infty}^{-\sqrt{\frac{E}{N_0}}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y^2\right\} dy = \text{erf}_*\left(-\sqrt{\frac{E}{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

With $\frac{E}{N_0} = 4 \Rightarrow p_e = \text{erf}_*(-2)$

$$= \frac{1}{2}\text{erfc}(\sqrt{2}) = 4 \times 10^{-2}$$

(a) Let Z denote the total observation space, which is divided into two parts Z_0 and Z_1. Whenever an observation falls in Z_0, we say H_0, and whenever an observation falls in Z_1, we say H_1. Thus, expressing the risk R in terms of the conditional probability density functions and the decision regions, we may write

$$
\begin{aligned}
R = & C_{00}\, p_0 \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)\,d\underline{x} \\
& + C_{10}\, p_0 \int_{Z_1} f_{\underline{X}|H_0}(\underline{x}|H_0)\,d\underline{x} \\
& + C_{11}\, p_1 \int_{Z_1} f_{\underline{X}|H_1}(\underline{x}|H_1)\,d\underline{x} \\
& + C_{01}\, p_1 \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)\,d\underline{x}
\end{aligned} \tag{1}
$$

For an N-dimensional observation space, the integrals in Eq. (1) are N-fold integrals.

To find the Bayes test, we must choose the decision regions Z_0 and Z_1 in such a manner that the risk R will be minimized. Because we require that a decision be made, this means that we must assign each point $\underline{x}$ in the observation space Z to Z_0 or Z_1; thus

$$Z = Z_0 + Z_1$$

Hence, we may rewrite Eq. (1) as

$$
\begin{aligned}
R = & p_0 C_{00} \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)\,d\underline{x} + p_0 C_{10} \int_{Z-Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)\,d\underline{x} \\
& + p_1 C_{11} \int_{Z-Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)\,d\underline{x} + p_1 C_{01} \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)\,d\underline{x}
\end{aligned} \tag{2}
$$

We observe that

$$\int_Z f_{\underline{X}|H_0}(\underline{x}|H_0)\,d\underline{x} = \int_Z f_{\underline{X}|H_1}(\underline{x}|H_1)\,d\underline{x} = 1$$

Hence, Eq. (2) reduces to

$$
\begin{aligned}
R = & p_0 C_{10} + p_1 C_{11} \\
& + \int_{Z_0} \{-[p_0(C_{10}-C_{00}) f_{\underline{X}|H_0}(\underline{x}|H_0)] + [p_1(C_{01}-C_{11}) f_{\underline{X}|H_1}(\underline{x}|H_1)]\}\,d\underline{x}
\end{aligned} \tag{3}
$$

The first two terms in Eq. (3) represent the fixed cost. The integral represents the cost controlled by those points $\underline{x}$ that we assign to Z_0. Since $C_{10} > C_{00}$ and $C_{01} > C_{11}$, we find that the two terms inside the square brackets are positive. Therefore, all values of $\underline{x}$ where the first term is larger than the second should be included in Z_0 because they contribute a negative amount to the integral. Similarly, all values of $\underline{x}$ where the second term is larger than the first should be excluded from Z_0 (i.e., assigned to Z_1) because they would contribute a positive amount to the integral. Values of $\underline{x}$ where the two terms are equal have no effect on the cost and may be assigned arbitrarily. Thus the decision regions are defined by the following statement: If

$$p_1(C_{01} - C_{11}) f_{\underline{X}|H_1}(\underline{x}|H_1) > p_0(C_{10} - C_{00}) f_{\underline{X}|H_0}(\underline{x}|H_0),$$

assign $\underline{x}$ to Z_1 and consequently say that H_1 is true. If the reverse is true, assign $\underline{x}$ to Z_0 and say H_0 is true.

Alternatively, we may write

$$\frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}$$

The quantity on the left is the likelihood ratio:

$$\Lambda(\underline{x}) = \frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)}$$

Let

$$\lambda = \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}$$

Thus, Bayes criterion yields a likelihood ratio test described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

(b) For the minimum probability of error criterion, the likelihood ratio test is described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0}{p_1}$$

Thus, we may view the minimum probability of error criterion as a special case of the Bayes criterion with the cost values defined as

$$C_{00} = C_{11} = 0$$

$$C_{10} = C_{01}$$

That is, the cost of a correct decision is zero, and the cost of an error of one kind is the same as the cost of an error of the other kind.