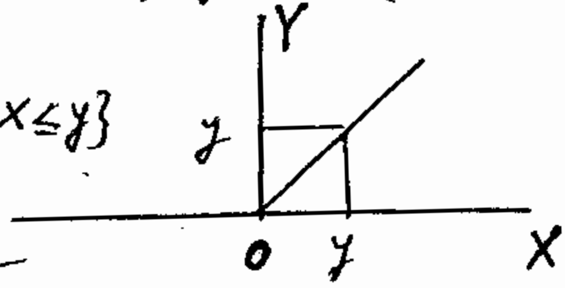
EECE 442 **Homework #1 Solution**

Problem #1 Let us first determine $F_Y(y) = P_r\{Y \le y\}$.



For $y>0$, $F_Y(y) = P_r\{Y \le y\} = P_r\{0 \le X \le y\}$

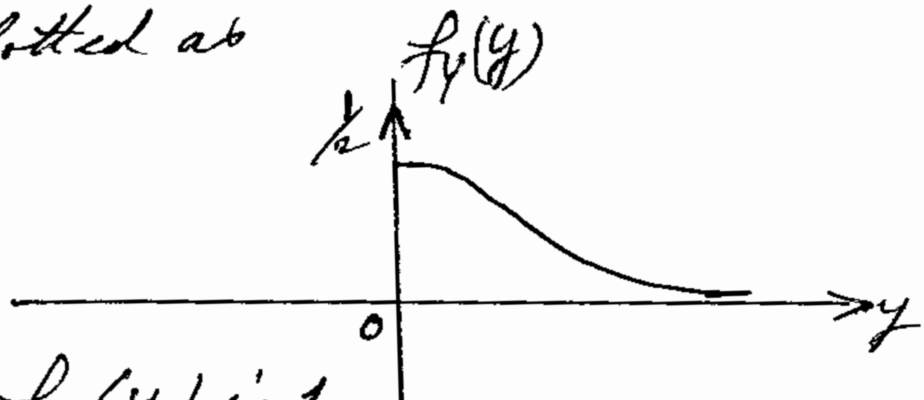
$$\Rightarrow F_Y(y) = \int_0^y \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_X^2}\right\} dx$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{1}{2}\frac{y^2}{\sigma_X^2}\right\}$$

And, $P_r\{Y=0\} = P_r\{X<0\} = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_X^2}\right\} dx = \frac{1}{2}$

$$\Rightarrow f_Y(y) = \begin{cases} 0 & , y<0 \\ \frac{1}{2}\delta(y) & , y=0 \\ \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{1}{2}\frac{y^2}{\sigma_X^2}\right\} & , y>0 \end{cases}$$

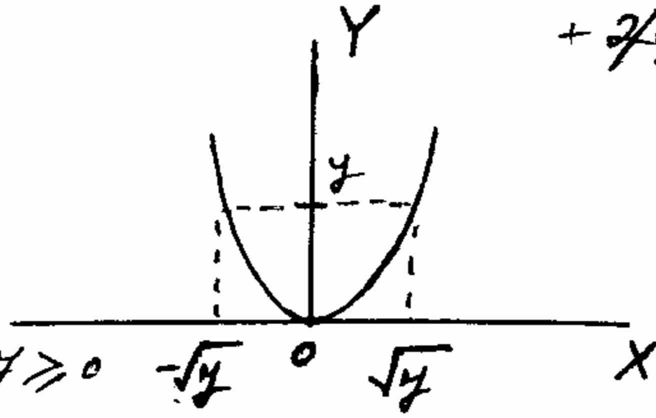
$f_Y(y)$ can be plotted as



The area under $f_Y(y)$ is 1.

+2/9

Problem #2



$$F_Y(y) = P_r\{Y \le y\}$$
$$= P_r\{-\sqrt{y} \le X \le \sqrt{y}\}, \quad y \ge 0$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_x^2}\right\} dx$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{1}{2}\frac{y}{\sigma_x^2}\right\} \frac{d}{dy}\sqrt{y}$$
$$- \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{1}{2}\frac{y}{\sigma_x^2}\right\} \frac{d}{dy}(-\sqrt{y})$$
$$= \frac{1}{\sqrt{2\pi y}\,\sigma_x} \exp\left\{-\frac{1}{2}\frac{y}{\sigma_x^2}\right\}$$

Thus, $f_Y(y) = \begin{cases} 0, & y < 0 \\ \dfrac{1}{\sqrt{2\pi y}\,\sigma_x} \exp\left\{-\dfrac{1}{2}\dfrac{y}{\sigma_x^2}\right\}, & y \ge 0 \end{cases}$

To check on whether the area under $f_Y(y) = 1$, let

$I = \int_0^\infty \frac{1}{\sqrt{2\pi y}\,\sigma_x} \exp\left\{-\frac{1}{2}\frac{y}{\sigma_x^2}\right\} dy$. Let $\frac{\sqrt{y}}{\sigma_x} = z$

$$\Rightarrow dz = \frac{1}{\sigma_x 2\sqrt{y}} dy$$

Since,

$$I = 2\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} dz = 1$$

Problem #3 $X(t) = A\cos(\omega_c t + \Theta)$

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

$$= E[A\cos(\omega_c t + \Theta) A\cos(\omega_c(t+\tau) + \Theta)]$$

$$= A^2 E[\cos(\omega_c t + \Theta)\cos(\omega_c(t+\tau) + \Theta)]$$

$$= \frac{A^2}{2} E[\cos(2\omega_c t + \omega_c\tau + 2\Theta) + \cos(\omega_c \tau)]$$

$$= \frac{1}{2}A^2\left[\int_0^{2\pi} \frac{1}{2\pi}\cos(2\omega_c t + \omega_c\tau + 2\theta)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\cos\omega_c\tau\,d\theta\right]$$

But $\cos(2\omega_c t + \omega_c\tau + 2\theta)$ is periodic in θ with period π. Hence $\int_0^{2\pi}\cos(2\omega_c t + \omega_c\tau + 2\theta)\,d\theta = 0$

$$\Rightarrow R_X(\tau) = \frac{1}{2}A^2\frac{1}{2\pi}\int_0^{2\pi}\cos\omega_c\tau\,d\theta$$

$$= \frac{1}{2}A^2\cos(\omega_c\tau)$$

$$R_X(0) = \frac{1}{2}A^2$$

$$m_X = E(X(t)) = \frac{1}{2\pi}\int_0^{2\pi}A\cos(\omega_c t + \theta)\,d\theta = 0$$

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\}$$

$$= \frac{1}{2}A^2\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

Problem # 4. $X(t) = A\cos(\omega_c t + \Theta)$

$$R_X(\tau) = E[X(t)X(t+\tau)]$$
$$= E[A\cos(\omega_c t + \Theta)\, A\cos(\omega_c(t+\tau) + \Theta)]$$
$$= E\left[\tfrac{1}{2}A^2\left(\cos(2\omega_c t + \omega_c\tau + 2\Theta) + \cos\omega_c\tau\right)\right]$$
$$= E\left[\tfrac{1}{2}A^2\cos(2\omega_c t + \omega_c\tau + 2\Theta)\right] + E\left[\tfrac{1}{2}A^2\cos\omega_c\tau\right]$$

Since A and Θ are statistically independent random variables, then

$$R_X(\tau) = \tfrac{1}{2}E(A^2)\,E(\cos(2\omega_c t + \omega_c\tau + 2\Theta)) + \tfrac{1}{2}\cos\omega_c\tau\, E(A^2).$$
$$= \frac{1}{2}\sigma_A^2\cos\omega_c\tau.$$

$$m_X = E\,X(t) = E[A\cos(\omega_c t + \Theta)]$$
$$= E[A]\,E[\cos(\omega_c t + \Theta)] = 0$$

$$K_X(\tau) = R_X(\tau) - m_X^2 = R_X(\tau) = \frac{1}{2}\sigma_A^2\cos\omega_c\tau$$

$$\tilde{S}_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \frac{1}{2}\sigma_A^2\,\pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)]$$

Problem # 5 $X(t) = A\cos\omega_c t$.

a) To show whether $X(t)$ is stationary or not, we need to check on whether

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = R_X(t_1 - t_2).$$

$$R_X(t_1, t_2) = E[A\cos\omega_c t_1\, A\cos\omega_c t_2]$$

$$= E\{[\tfrac{1}{2}A^2][\cos(\omega_c(t_1+t_2)) + \cos\omega_c(t_1-t_2)$$

$$= \frac{1}{2}\sigma_A^2[\cos\omega_c(t_1+t_2) + \cos\omega_c(t_1-t_2)]$$

Hence, $X(t)$ is non-stationary.

b) $Y(T) = \int_0^T X(z)\,dz$.

$Y(T)$ is a Gaussian random Variable.

$$E(Y) = E\int_0^T A\cos\omega_c z\,dz = 0$$

$$\sigma_Y^2 = E(Y^2) = E\left[\int_0^T A\cos(\omega_c t)\,dt \int_0^T A\cos(\omega_c u)\,du\right]$$

$$= E[A^2]\int_0^T \cos\omega_c t\,dt \int_0^T \cos\omega_c u\,du$$

$$= \sigma_A^2 \times \left[\frac{1}{\omega_c}\sin\omega_c t\Big|_0^T\right]^2 = \sigma_A^2\frac{1}{\omega_c^2}\sin^2(\omega_c T)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y}\exp\left\{-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right\}$$

With $\sigma_Y^2 = \frac{\sigma_A^2}{\omega_c^2}\sin^2(\omega_c T)$.

c) $R_X(t_1,t_2) = E[Y(t_1)\,Y(t_2)]$

$$= E\left[\int_0^{t_1} A\cos\omega_c\tau\,d\tau \int_0^{t_2} A\cos\omega_c\tau\,d\tau\right]$$

$$= E(A^2)\int_0^{t_1}\cos\omega_c\tau\,d\tau \int_0^{t_2}\cos\omega_c\tau\,d\tau$$

$$= \sigma_A^2\,\frac{1}{\omega_c}\sin\omega_c\tau\Big|_0^{t_1} \times \frac{1}{\omega_c}\sin\omega_c\tau\Big|_0^{t_2}$$

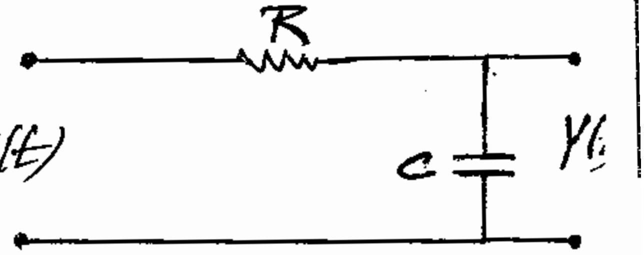
$$= \sigma_A^2\,\frac{1}{\omega_c}\sin\omega_c t_1 \times \frac{1}{\omega_c}\sin\omega_c t_2$$

$$= \frac{\sigma_A^2}{\omega_c^2}\sin\omega_c t_1\,\sin\omega_c t_2$$

$$= \frac{\sigma_A^2}{2\omega_c^2}\left[\cos\omega_c(t_1-t_2) - \cos\omega_c(t_1+t_2)\right]$$

Thus, $R_X(t_1,t_2)$ is not a function of (t_1-t_2) only and the random process $Y(t)$ is non-stationary.

Problem #6



$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$$

$$H(\omega) = \frac{1}{1+j\omega RC}$$

$$\Rightarrow |H(\omega)|^2 = \frac{1}{1+(\omega RC)^2}.$$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\,e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty} e^{-2\gamma|\tau|}\,e^{-j\omega\tau}\,d\tau$$

$$S_X(\omega) = \int_{-\infty}^{0} e^{+2\nu\tau} e^{-j\omega\tau} d\tau + \int_0^{\infty} e^{-2\nu\tau} e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2\nu - j\omega)\tau} d\tau + \int_0^{\infty} e^{-(2\nu + j\omega)\tau} d\tau$$

$$= \frac{e^{(2\nu - j\omega)\tau}}{2\nu - j\omega}\Big|_{-\infty}^{0} - \frac{e^{-(2\nu + j\omega)\tau}}{2\nu + j\omega}\Big|_0^{\infty}$$

$$= \frac{1}{2\nu - j\omega} + \frac{1}{2\nu + j\omega} \quad \text{for } \underline{\underline{\nu > 0}}$$

$$= \frac{4\nu}{4\nu^2 + \omega^2}$$

$$\Rightarrow S_Y(\omega) = \frac{4\nu}{(4\nu^2 + \omega^2)(1 + \omega^2 (RC)^2)}$$

$$R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4\nu}{(4\nu^2 + \omega^2)(1 + \omega^2 (RC)^2)} e^{j\omega\tau} d\omega$$

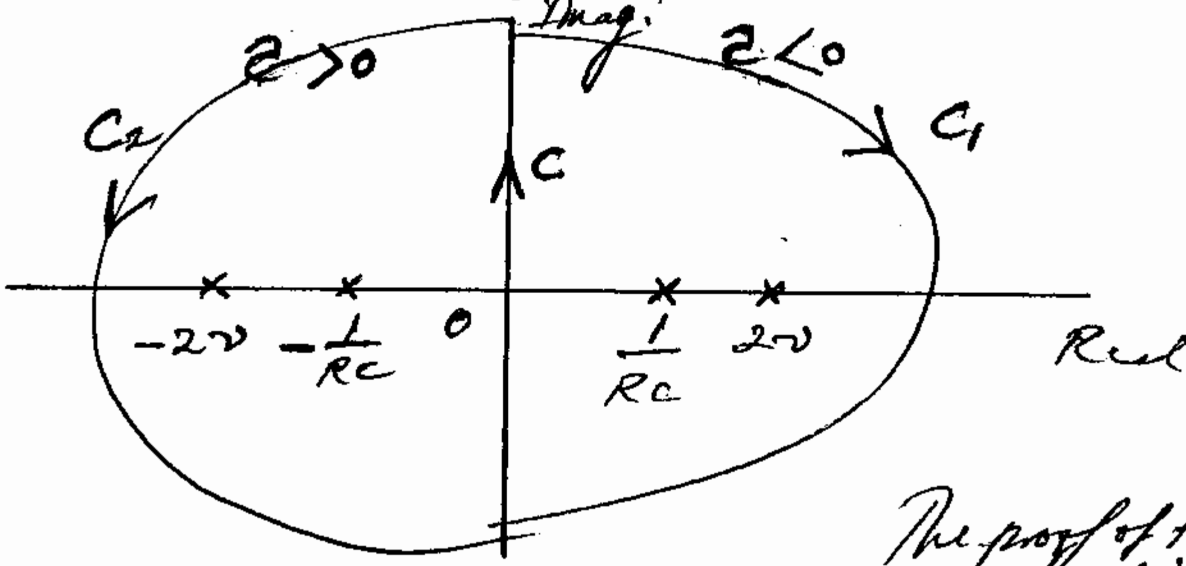
This integral can be evaluated using the integral tables or complex integration.

Let $j\omega = z \Rightarrow d\omega = -j\,dz$

$$R_Y(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{4\nu/(RC)^2 \, e^{z\tau}}{(4\nu^2 - z^2)\left(\left(\frac{1}{RC}\right)^2 - z^2\right)} dz$$

$$= \frac{1}{2\pi j} \int_C \frac{4\nu/(RC)^2 \quad e^{z\tau}}{(z^2 - 4\nu^2)\left(z^2 - \left(\frac{1}{RC}\right)^2\right)} dz$$

The contour C is the imaginary axis in the z-plane



Z-plane

The proof of this statement is beyond the scope of this course

The integration along c_1 and c_2 is 0. Thus,

for $\tau > 0$
$$R_y(\tau) = \frac{1}{2\pi j}\oint_{C+C_2} \frac{4v/(RC)^2 e^{z\tau}}{(z^2-4v^2)(z^2-(\frac{1}{RC})^2)} dz$$

$R_y(\tau) = \sum$ residues at poles inside $(C+C_2)$.

$$= \left.\frac{4v/(RC)^2 e^{z\tau}}{(z-2v)(z^2-(\frac{1}{RC})^2)}\right|_{z=2v} + \left.\frac{4v/(RC)^2 e^{z\tau}}{(z^2-4v^2)(z-\frac{1}{RC})}\right|_{z=-\frac{1}{RC}}$$

$$= \frac{1}{(1-(2vRC)^2)} e^{-2v\tau} - \frac{2vRC}{1-(2vRC)^2} e^{-\frac{1}{RC}\tau}$$

$$= \frac{1}{(1-(2vRC)^2)}\left[e^{-2v\tau} - 2vRC\, e^{-\frac{1}{RC}\tau}\right]$$

For $\tau < 0$ the integration is along $C + c_1$.

$$\therefore R_y(\tau) = -\sum \text{residues at poles inside } C + c_1.$$

$$\therefore R_y(\tau) = \frac{1}{(1-(2\nu RC)^2)}\left[e^{2\nu\tau} - 2\nu RC\, e^{+\frac{1}{RC}\tau}\right]$$

This could be determined directly by noticing that $R_y(\tau)$ is even, i.e., $R_y(\tau) = R_y(-\tau)$.

With $2\nu RC = k \Rightarrow$

$$R_y(\tau) = \begin{cases} \dfrac{1}{1-k^2}\left(e^{-\frac{k}{RC}\tau} - k\, e^{-\frac{1}{RC}\tau}\right), & \tau > 0 \\[2ex] \dfrac{1}{1-k^2}\left(e^{\frac{k}{RC}\tau} - k\, e^{\frac{1}{RC}\tau}\right), & \tau < 0 \\[2ex] \dfrac{1}{1+k} & , \tau = 0 \end{cases}$$

Hence $R_y(0) = \dfrac{1}{1+k} = \dfrac{1}{1+2\nu RC}$ and this is the filter output power.