# EECE 442 Communication Systems

# Problems and Solutions — Chapter II - Analog modulation

**Problem 1**

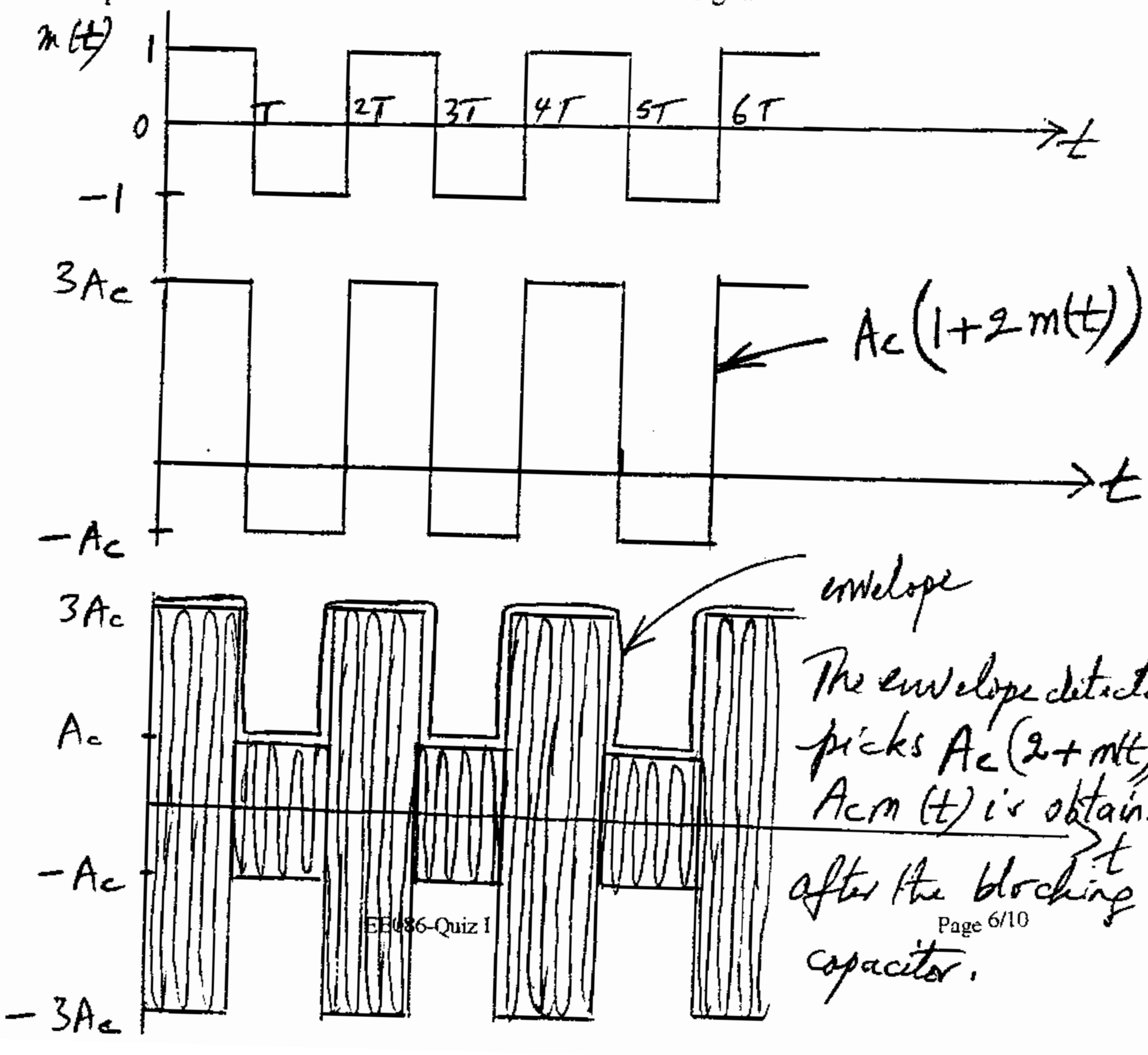
Consider the following DSBLC (AM) signal s(t)

$$s(t) = A_c(1 + 2m(t))\cos(\omega_c t)$$

where

$$m(t) = \begin{cases} 1 & 2kT \le t \le (2k+1)T, \quad k = 0,1,2,\ldots \\ -1 & (2k-1)T \le t \le 2kT \quad k = 1,2,3,\ldots \end{cases}$$

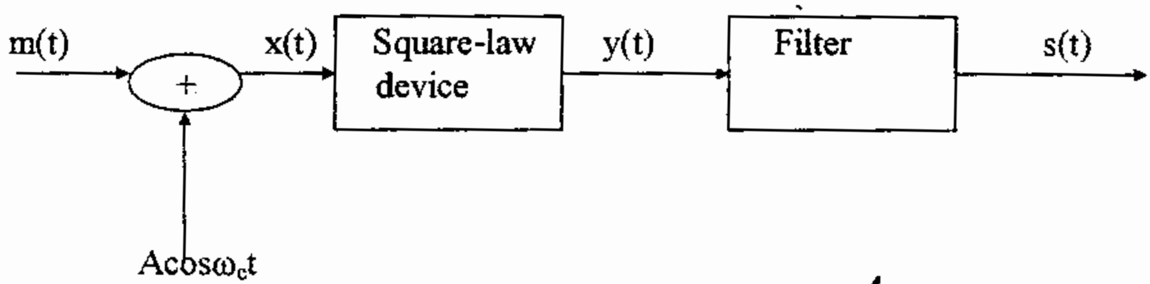
T is a positive constant much greater than $\frac{1}{f_c}$. Plot the envelope of s(t) and conclude whether or not an envelope detector followed by a blocking capacitor can be used to demodulate the DSB-LC signal.




The envelope detector picks $A_c(2+m(t))$ $A_c m(t)$ is obtained after the blocking capacitor.

**Problem 2**

Consider the system shown below




$m(t)$ is a sinusoidal signal: $m(t) = M\cos(\omega_m t)$.

Assume that [redacted] Also assume that the square-law device is defined by: $y(t) = 2x(t) + 4x^2(t)$.

a. Write the equation of y(t). $m(t) = M\cos(\omega_m t)$

$$y(t) = 2[m(t) + A\cos\omega_c t] + 4[m(t) + A\cos(\omega_c t)]^2$$

$$= 2m(t) + 4m^2(t) + 2A^2 + 2A\cos\omega_c t + 8m(t)\cos\omega_c t + 2A^2\cos(2\omega_c$$

b. Describe the filter which yields a DSB-LC (AM) signal at the output. Give the necessary filter type and the frequencies of interest.

Using a BPF with center frequency $\omega_c \gg 3\omega_m$ and bandwidth $2\omega_m$, we obtain

$$s(t) = 2A\cos(\omega_c t) + 8A\,m(t)\cos(\omega_c t) = 2A[1 + 4m(t)]\cos(\omega_c t)$$

c. What value of M yields a modulation index of 0.1?

$$\mu = 4M = 0.1 \Rightarrow M = \frac{0.1}{4} = 0{,}025$$

**Problem 3**

Consider the DSB-LC signal

$$S(t) = A_c[1 + \cos(\omega_m t)\cos(2\omega_m t)]\cos(\omega_c t)$$

where $\omega_c >> \omega_m$. Let s(t) be inputted to an ideal band-pass filter centred at $\omega_c$ and bandwidth $4\omega_m$ rad/s.

1. Determine the signal at the output of the band-pass filter.

$$s(t) = A_c\left[1 + \frac{1}{2}\cos\omega_m t + \frac{1}{2}\cos(3\omega_m t)\right]\cos(\omega_c t)$$

$$= A_c\left[1 + \frac{1}{2}\cos(\omega_m t)\right]\cos(\omega_c t) + \frac{A_c}{2}\cos(3\omega_m t)\cos(\omega_c t)$$

The BPF picks $A_c\left[1 + \frac{1}{2}\cos(\omega_m t)\right]\cos(\omega_c t)$

2. Can an envelope detector be used to demodulate the signal determined in (1)? If yes, determine the output of this envelope detector.

An envelope detector can be used to demodulate the signal in (1) since it has 50% percentage modulation. The envelope detector output is:

$$A_c\left(1 + \frac{1}{2}\cos(\omega_m t)\right)$$

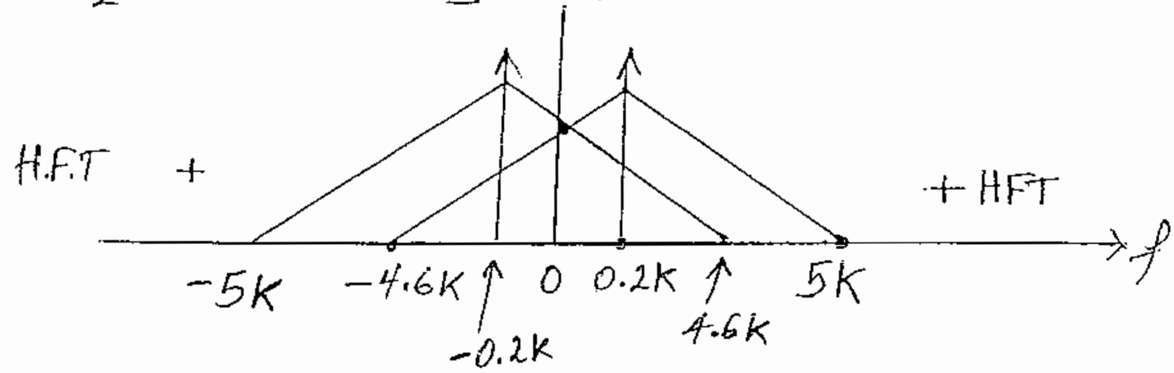
**Problem 4**

A message signal m(t) of bandwidth 4.8 KHz is DSB-LC (AM) modulated with a carrier frequency of 1500 KHz. Detection is done using a coherent detector with frequency shift of 0.2 KHz. Determine the bandwidth of the signal at the output of the receiver given that the bandwidth of low pass filter in the coherent detector is 10 KHz.

The output of the product modulator is:

$$A_c[1+k_a m(t)]\cos\omega_c t \times \cos(\omega_c+\Delta\omega)t$$

$$= \frac{1}{2}A_c[1+k_a m(t)]\cos(\Delta\omega t) + H.F.T$$




The LPF output is: $\frac{A_c}{2}[1+k_a m(t)]\cos(\Delta\omega t)$

It has a bandwidth of 5 KHz.

**Problem 5**

A 1KHz sinusoid is DSB-LC modulated onto a carrier so that 100% modulation results. At the receiver, the audio signal is recovered by a band-pass filter followed by an envelope detector. Suppose that due to mistuning, the upper-sideband of the AM signal is completely eliminated, but the carrier and one sideband are not affected.

a. Determine the output of the band-pass filter.

$$S(t) = A_c[1+\cos(\omega_m t)]\cos(\omega_c t);$$ DSB-LC with 100% modulation.

$$= \underbrace{A_c \cos \omega_c t}_{\text{Carrier}} + \underbrace{\frac{A_c}{2}\cos(\omega_c - \omega_m)t}_{\text{Lower}} + \underbrace{\frac{A_c}{2}\cos(\omega_c+\omega_m)t}_{\text{upper}}$$

The BPF output is: $A_c \cos(\omega_c t) + \frac{A_c}{2}\cos(\omega_c - \omega_m)t$.

a. Determine the output of the envelope detector.

$$y = A_c\cos(\omega_c t) + \frac{A_c}{2}\cos(\omega_c t)\cos(\omega_m t) + \frac{A_c}{2}\sin(\omega_c t)\sin(\omega_m t)$$

$$= \left[A_c + \frac{A_c}{2}\cos\omega_m t\right]\cos(\omega_c t) + \frac{A_c}{2}\sin(\omega_m t)\sin(\omega_c t)$$

The output of the envelope detector is:

$$[y(t)]_{env} = \left[A_c^2 + \frac{A_c^2}{4}\cos^2\omega_m t + A_c^2\cos\omega_m t + \frac{A_c^2}{4}\sin^2\omega_m t\right]^{1/2}$$

$$= \left[\frac{5A_c^2}{4} + A_c^2\cos\omega_m t\right]^{1/2}$$

$$= A_c\left[\frac{5}{4} + \cos(\omega_m t)\right]^{1/2}$$

**Problem 6**

Consider the following signal: $s(t) = 0.2\cos\omega_0 t + 10\cos\omega_1 t + 0.2\cos\omega_2 t$, where $\omega_0 = 500\times10^3$ rad/s, $\omega_1 = 501\times10^3$ rad/s, and $\omega_2 = 502\times10^3$ rad/s. What will be the output if s(t) is applied to:

a. A synchronous detector which is synchronized to the highest frequency component of s(t). The local oscillator output has unit amplitude and the bandwidth of the LPF is 3 KHz.

The output of the product modulator is:

$$= 0.2\cos(\omega_2 t)\cos(\omega_0 t) + 10\cos(\omega_2 t)\cos(\omega_1 t) + 0.2\cos^2(\omega_2 t)$$

$$= 0.1\cos(2\times10^3 t) + 0.1\cos(1002\times10^3 t) + 5\cos(10^3 t) +$$

$$5\cos(1003\times10^3 t) + 0.1 + 0.1\cos(1004\times10^3 t)$$

The LPF output is: $0.1 + 5\cos(10^3 t) + 0.1\cos(2\times10^3 t)$.

b. An envelope detector followed by a blocking capacitor.

$$s(t) = 10\left[1 + \frac{0.02\left(\cos(500\times10^3 t) + \cos(502\times10^3 t)\right)}{\cos(501\times10^3 t)}\right] \cos(501\,10^3 t)$$

$$= 10\left[1 + \frac{0.02\times2\cos(501\,10^3 t)\cos(10^3 t)}{\cos(501\,10^3 t)}\right]\cos(501\,10^3 t)$$

$$= 10\left[1 + 0.04\cos(10^3 t)\right]\cos(501\,10^3 t)$$

The envelope detector output is $(10 + 0.4\cos(10^3 t))$

The blocking capacitor output is $0.4\cos(10^3 t)$

**Problem 7**

A frequency discriminator detector followed by an integrator is used to demodulate a phase-modulated signal $s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$. Determine the output of this demodulator.

The differentiator output is:

$$-A_c\left[\omega_c + k_p \frac{dm(t)}{dt}\right]\sin(\omega_c t + k_p m(t))$$

The output of the envelope detector is:

$$A_c\left[\omega_c + k_p \frac{dm(t)}{dt}\right]$$

The blocking capacitor output is:

$$A_c k_p \frac{dm(t)}{dt}$$

The integrator output is:

$$A_c k_p m(t).$$

**Problem 8**

Determine the maximum value of the instantaneous frequency of the FM signal whose modulating signal is given by: $m(t) = [-100\sin(20\pi t) - 50\sin(60\pi t)]\mathrm{V}$. Let the carrier frequency be 92 MHz and the frequency sensitivity of the modulator be 100 Hz/V.

$$s(t) = A_c \cos\left(\omega_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_c + 2\pi k_f\, m(t)$$

$$= 2\pi \times 92 \times 10^6 + 2\pi \times 100 \times 50\ (-2\sin(20\pi t) - \sin(6$$

$$f_i(t) = 92 \times 10^6 - 5000\left(2\sin(20\pi t) + \sin(60\pi t)\right)$$

The maximum value of $f_i(t)$ is attained when $[2\sin(20\pi t) + \sin(60\pi t)]$ reaches its minimum.

The minimum is $-2.1$.

$$\Rightarrow \max f_i(t) = 92 \times 10^6 + 10500 \text{ Hz}.$$

**Problem 9**

Consider the following Frequency modulated signal s(t)

$$s(t) = 4\cos(9195t + 31 + 100\sin(90t) + 50\sin(60t) + 50\sin(30t))$$

a. Determine the carrier frequency in rad/sec.

$$\omega_c = 9195 \text{ rad/s}$$

b. Determine the maximum frequency deviation in rad/sec.

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = 9195 + 9000\cos(90t) + 3000\cos(\text{(}$$
$$+ 1500\cos(30t)$$

$$\Delta\omega = 9000 + 3000 + 1500 = 13500 \text{ rad/s}$$

c. Determine the bandwidth of the message signal in rad/sec.

$$m(t) = \frac{1}{2\pi k_f}\left[9000\cos(90t) + 3000\cos(60t) + 1500\cos(30t)\right]$$

$$\Rightarrow \text{BW of } m(t) = 90 \text{ rad/s}.$$

**Problem 10**

Consider a single tone FM signal with amplitude 10 Volts, frequency 5 KHz, and modulation index $\beta = 2$. Determine the ratio of the average power of the frequency components of this FM signal contained within its bandwidth to the total signal average power. Use Carson's rule for bandwidth computation.

$J_0(2) = 0.2239, J_1(2) = 0.5767, J_2(2) = 0.3528, J_3(2) = 0.1289,$

$J_4(2) = 0.0340, J_5(2) = 0.007, J_6(2) = 0.0012$

Carson's rule considers side frequencies around $\omega_c$ up to $\beta + 1 = 3$,

$\Rightarrow$ Average power in these side frequencies is:

$$P = \sum_{n=-3}^{3} \frac{1}{2} A_c^2 J_n^2(\beta)$$

Since $s(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(\omega_c + n\omega_m)t$

$$\Rightarrow R = \frac{P}{\frac{1}{2} A_c^2} = \sum_{n=-3}^{3} J_n^2(2)$$

$$= (0.2239)^2 + 2\left[(0.5767)^2 + (0.3528)^2 + (0.1289)^2\right]$$

$$= 0.99746$$

**Problem 11**

A voltage controlled oscillator (VCO) is used to generate an FM signal. When a single tone $a\cos(2\pi f_1 t)$ is inputted to this VCO, the output FM signal has a bandwidth of 20 KHZ. Another tone signal $2a\cos(2\pi f_2 t)$ is next inputted to this VCO, the resulting FM signal has a bandwidth of 40 KHz. Determine $\frac{f_1}{f_2}$.

$$S_1(t) = A_c \cos\left(\omega_c t + \frac{2\pi k_f a}{\omega_1} \sin \omega_1 t\right)$$

$$S_2(t) = A_c \cos\left(\omega_c t + \frac{2\pi k_f 2a}{\omega_2} \sin \omega_2 t\right)$$

$$B_{T_1} = 2(2\pi k_f a + \omega_1) = 20 \quad ①$$

$$B_{T_2} = 2(2\pi k_f 2a + \omega_2) = 40 \quad ②$$

$$① \Rightarrow 2\pi k_f a = 10 - \omega_1$$

$$② \Rightarrow 2(10 - \omega_1) + \omega_2 = 20$$

$$\Rightarrow -2\omega_1 + \omega_2 = 0$$

$$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{1}{2}$$

**Problem 12**

Determine the output signal-to-noise ratio of an envelope detector whose input is given by: $r(t) = A_c \cos[2\pi f_c t + \phi(t)] + n(t)$. $n(t)$ is a band-pass noise centred about $f_c$ of bandwidth 2B and of power spectral density $N_0/2$ Watts/Hz. Assume high carrier to noise ratio and note that the envelope detector is not followed by a blocking capacitor.

$$r(t) = A_c \cos(\omega_c t + \phi(t)) + n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

$$= A_c \cos\omega_c t \cos\phi(t) - A_c \sin\omega_c t \sin\phi(t) + n_c(t)\cos\omega_c t - n_s(t)\sin(\omega_c t)$$

$$= [A_c \cos\phi(t) + n_c(t)]\cos\omega_c t - [A_c \sin\phi(t) + \ldots]\sin(\omega_c t)$$

$$[r(t)]_{env.} = \left[(A_c\cos\phi + n_c)^2 + (A_c \sin\phi + n_s)^2\right]^{1/2}$$

$$= \left[A_c^2 + 2A_c\cos\phi\, n_c + 2A_c \sin\phi\, n_s + n_c^2 + n_s^2\right]$$

$$\cong A_c\left[1 + \frac{2n_c\cos\phi + 2n_s\sin\phi}{A_c}\right]^{1/2}$$

$$\cong A_c + (n_c\cos\phi + n_s \sin\phi) = A_c + n_0(t)$$

$$P_S = A_c^2, \quad P_{n_0} = E[n_0^2(t)] = E[n_c^2(t)]$$

$$= \frac{N_0 W}{\pi}$$

$$\Rightarrow \frac{S}{N} = \frac{\pi A_c^2}{N_0 W} = \frac{A_c^2}{2N_0 B}.$$

**Problem 13**

A DSB-LC signal of the form $s(t) = A_c[1+k_a m(t)]\sin(\omega_c t + \phi)$ is present along with white noise at the input of an AM receiver that consists of a band-pass filter of bandwidth 2B Hz, an envelope detector, a blocking capacitor, and a low-pass filter of bandwidth B Hz.. Determine the maximum output signal to noise ratio assuming large carrier-to-noise ratio. Note that B is the bandwidth of message signal m(t).

The input of the envelope detector is:

$$x(t) = [A_c(1+k_a m(t))\sin\phi + n_c(t)]\cos\omega_c t + [A_c(1+k_a m(t))\cos\phi - n_s(t)]\sin\omega_c t$$

Envelope detector output is:

$$A_c(1+k_a m(t)) + n_c(t)\sin\phi - n_s(t)\cos\phi$$

Blocking capacitor output is:

$$A_c k_a m(t) + \underbrace{n_c(t)\sin\phi - n_s(t)\cos\phi}_{n_o(t)}$$

$$E(n_o^2(t)) = E(n_c^2(t)) = 2N_o B$$

$$E_s = A_c^2 k_a^2 P$$

$$\Rightarrow \frac{S}{N} = \frac{A_c^2 k_a^2 P}{2N_o B}.$$

**Problem 14**

A DSB-SC signal is present along with white noise at the input of a synchronized receiver that consists of a band-pass filter of mid-frequency $f_c$, a mixer, and a low-pass filter. The local oscillator is synchronized in frequency and phase with the incoming signal. Determine the output signal-to-noise ratio.

The product modulator input is:

$$x(t) = A_c m(t)\cos\omega_c t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t.$$

$$= [A_c m(t) + n_c(t)]\cos\omega_c t - n_s(t)\sin\omega_c t$$

The product modulator output is:

$$\frac{1}{2}[A_c m(t) + n_c(t)][1+\cos(2\omega_c t)] - \frac{1}{2} n_s(t)\sin(2\omega_c$$

the LPF output is:

$$\frac{1}{2}[A_c m(t) + n_c(t)]$$

$$P_s = \frac{1}{4}A_c^2 P \,, \quad P_{n_c} = \frac{1}{4} \times \frac{N_0 W}{\pi}$$

$$\Rightarrow \frac{S}{N} = \frac{\pi A_c^2 P}{N_0 W}$$