EECE442 Midterm Solution Summer 07

Problem #1

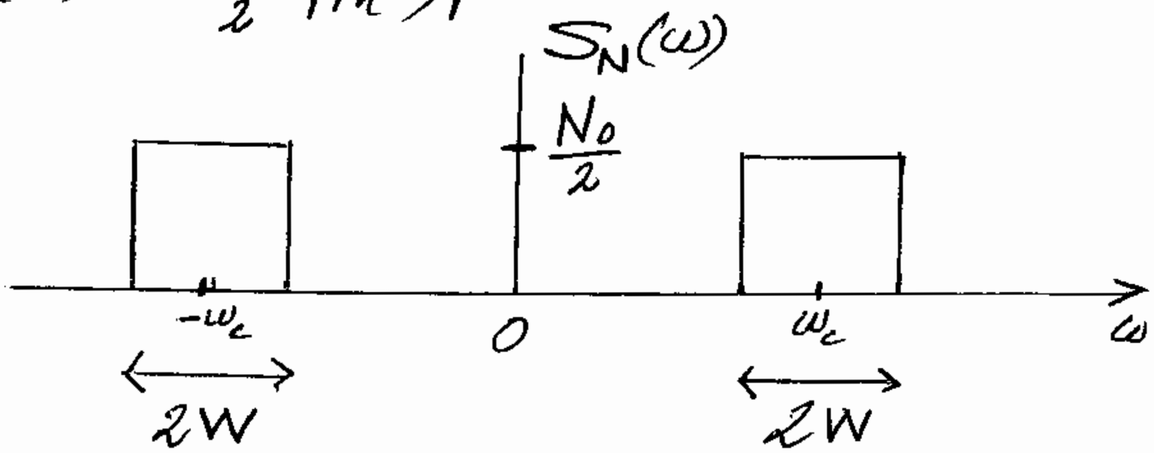
$$\sigma_N^2 = E[(N(t) - \bar{N})^2],$$

where t is any time instant at which the stationary process $N(t)$ is sampled.

$\bar{N}(t) = 0$ since the input process, $W(t)$, has a mean equal to 0. Hence,

$$\sigma_N^2 = E[N^2(t)] = R_N(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\,d\omega.$$

$$S_N(\omega) = \frac{N_0}{2}|H(\omega)|^2$$



$$\sigma_N^2 = \frac{1}{2\pi} \times \frac{N_0}{2} \times 2W \times 2 = \frac{N_0 W}{\pi}$$

σ_N^2 is also the average power of the noise process available at the output of the ideal BPF.

Problem #2

$$R_N(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_c-W}^{-\omega_c+W} \frac{N_0}{2} e^{j\omega\tau} d\omega + \int_{\omega_c-W}^{\omega_c+W} \frac{N_0}{2} e^{j\omega\tau} d\omega\right]$$

$$= \frac{N_0}{4\pi}\left[\frac{1}{j\tau} e^{j\omega\tau}\Big|_{-\omega_c-W}^{-\omega_c+W} + \frac{1}{j\tau} e^{j\omega\tau}\Big|_{\omega_c-W}^{\omega_c+W}\right]$$

$$= \frac{N_0}{4\pi}\left[\frac{2e^{-j\omega_c\tau}}{\tau}\left[\frac{e^{jW\tau} - e^{-jW\tau}}{2j}\right] + \frac{2e^{j\omega_c\tau}}{\tau}\left[\frac{e^{jW\tau} - e^{-jW\tau}}{2j}\right]\right]$$

$$= \frac{N_0 \times 4}{4\pi}\frac{\sin(W\tau)}{\tau}\left[\frac{e^{j\omega_c\tau} + e^{-j\omega_c\tau}}{2}\right]$$

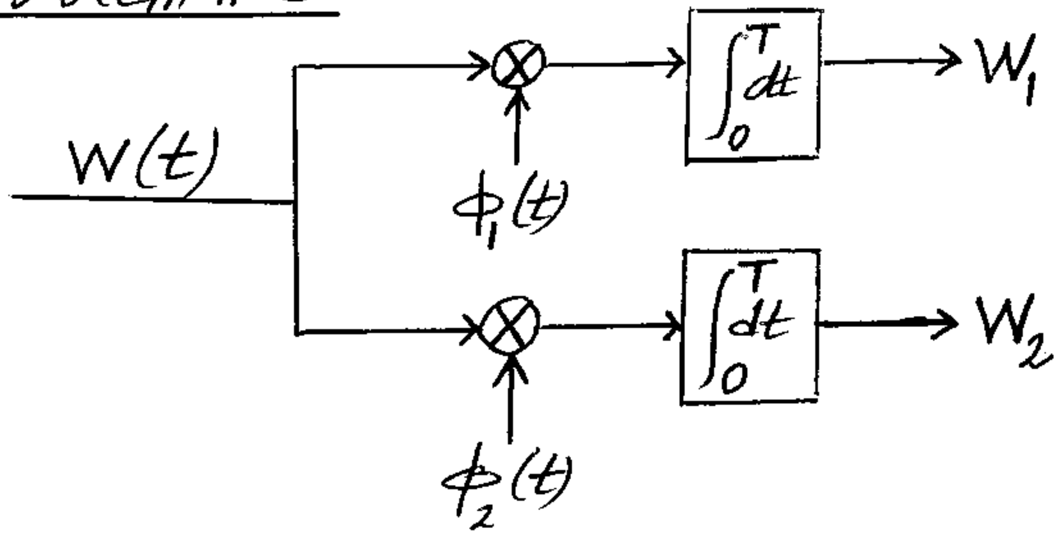
$$R_N(\tau) = \frac{N_0}{\pi\tau}\sin(W\tau)\cos(\omega_c\tau)$$

$$= \frac{N_0 W}{\pi}\,\frac{\sin(W\tau)}{(W\tau)}\cos(\omega_c\tau)$$

$$R_N(\tau) \longrightarrow \frac{N_0 W}{\pi} \text{ when } \tau \to 0.$$

This is consistent with the result in Problem 1.

Problem #3



$$\mathrm{Cov}(W_1, W_2) = E\left[(W_1 - \overline{W}_1)(W_2 - \overline{W}_2)\right]$$

$$\overline{W}_1 = E(W_1) = E\left[\int_0^T w(t)\phi_1(t)\,dt\right]$$

$$= \int_0^T \underbrace{E[w(t)]}_{0}\phi_1(t)\,dt = 0$$

Also, $E(W_2) = 0$.

Hence, $\mathrm{Cov}(W_1 W_2) = E[W_1 W_2]$

$$= E\left[\int_0^T w(t)\,\phi_1(t)\,dt \int_0^T w(u)\,\phi_2(u)\,du\right]$$

$$= \int_0^T \phi_1(t)\,dt \int_0^T \underbrace{E[w(t)\,w(u)]}_{\frac{N_0}{2}\delta(t-u)}\phi_2(u)\,du$$

$$= \frac{N_0}{2}\int_0^T \phi_1(t)\phi_2(t)\,dt \underbrace{\int_0^T \delta(t-u)\,du}_{1}$$

$$= \frac{N_0}{2}\int_0^T \phi_1(t)\phi_2(t)\,dt = 0$$

Problem # 4

$$\sigma_{W_1}^2 = E\left[(W_1 - \bar{W}_1)^2\right] = E(W_1^2)$$

$$= E\left[\int_0^T w(t)\phi_1(t)\,dt \int_0^T w(u)\phi_1(u)\,du\right]$$

$$= \int_0^T \phi_1(t)\,dt \int_0^T \underbrace{E[w(t)w(u)]}_{\frac{N_0}{2}\delta(t-u)} \phi_1(u)\,du$$

$$= \int_0^T \frac{N_0}{2}\phi_1^2(t)\,dt \underbrace{\int_0^T \delta(t-u)\,du}_{1}$$

$$= \frac{N_0}{2}\underbrace{\int_0^T \phi_1^2(t)\,dt}_{1} = \frac{N_0}{2} = \sigma_{W_2}^2.$$

Problem #5

$$x(t) = A_c m(t)\cos(\omega_c t + \phi) + n(t)$$

$$x(t) = A_c m(t)\cos\phi\cos(\omega_c t) - A_c m(t)\sin\phi\,\sin(\omega_c t)$$

$$+ n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

$$= \left[A_c m(t)\cos\phi + n_c(t)\right]\cos(\omega_c t)$$

$$- \left[A_c m(t)\sin\phi + n_s(t)\right]\sin(\omega_c t)$$

$$y(t) = Env(x(t))$$

$$= \left\{ \left[A_c m(t) \cos\phi + n_c(t)\right]^2 + \left[A_c m(t) \sin\phi + n_s(t)\right]^2 \right\}^{1/2}$$

$$= \left\{ A_c^2 m^2(t) \cos^2\phi + n_c^2(t) + 2A_c m(t) n_c(t) \cos\phi + A_c^2 m^2(t) \sin^2\phi + n_s^2(t) + 2A_c m(t) n_s(t) \sin\phi \right\}^{1/2}$$

$$= \left\{ A_c^2 m^2(t) + 2A_c m(t)\left[n_c(t)\cos\phi + n_s(t)\sin\phi\right] + n_c^2(t) + n_s^2(t) \right\}^{1/2}$$

$$= A_c |m(t)| \left\{ 1 + \frac{2(n_c(t)\cos\phi + n_s(t)\sin\phi)}{A_c m(t)} + \frac{n_c^2(t) + n_s^2(t)}{A_c^2 m^2(t)} \right\}^{1/2}$$

Since $m(t)$ is always positive and the carrier is very large compared to the noise, then

$$y(t) \simeq A_c m(t) \left\{ 1 + \frac{2\left[n_c(t)\cos\phi + n_s(t)\sin\phi\right]}{A_c m(t)} \right\}^{1/2}$$

$$\simeq A_c m(t) \left\{ 1 + \frac{n_c(t)\cos\phi + n_s(t)\sin\phi}{A_c m(t)} \right\}$$

$$= \underbrace{A_c m(t)}_{S_o(t)} + \underbrace{n_c(t)\cos\phi + n_s(t)\sin\phi}_{n_o(t)}$$

Problem #6

If $\phi = \pi/2$, then the output of the envelope detector becomes:

$$y(t) = A_c m(t) + n_s(t)$$

$$SNR = \frac{A_c^2 E(m^2(t))}{E[n_s^2(t)]} = \frac{A_c^2 P}{\frac{N_0 W}{\pi}}$$

$$= \frac{\pi A_c^2 P}{N_0 W}$$

Problem #7

$$s(t) = A_c\left(1 + \mu\cos(\omega_m t)\right)\cos(\omega_c t)$$

$$A_{max} = A_c(1+\mu); \quad A_{min} = A_c(1-\mu)$$

$$\Rightarrow \mu = \frac{A_{max} - A_{min}}{A_{max} + A_{min}} \qquad \text{for } \mu > 0.$$

$$= \frac{2.25 A_c + 0.25 A_c}{2.25 A_c - 0.25 A_c} = \frac{2.5}{2} = 1.25$$

For $\mu < 0$, $A_{max} = A_c(1-\mu)$; $A_{min} = A_c(1+\mu)$

$$\Rightarrow \mu = \frac{A_{min} - A_{max}}{A_{min} + A_{max}} = -1.25$$

Percentage modulation $= |\mu| \times 100\% = 125\%$

Problem # 8

$$s(t) = \frac{A_c}{2}\left[1 + \frac{4}{\pi A_c} m(t)\right]\cos(\omega_c t)$$

$$\Rightarrow s(t) = \underbrace{\frac{A_c}{2}\cos(\omega_c t)}_{\text{Carrier}} + \underbrace{\frac{2}{\pi} m(t)\cos(\omega_c t)}_{\text{DSB-SC}}$$

$$P_c = \frac{1}{2}\left(\frac{A_c}{2}\right)^2 = \frac{A_c^2}{8}$$

$$P_{DSB-SC} = \frac{4}{\pi^2} E\left[m^2(t)\cos^2(\omega_c t)\right]$$

$$= \frac{2}{\pi^2}\left[E(m^2(t)) + \cancel{E(m^2(t)\cos(2\omega_c t))}^{\,0}\right]$$

$$= \frac{2P}{\pi^2}$$

$$\frac{P_c}{P_{DSB-SC}} = \frac{\pi^2 A_c^2}{16P}$$

Problem #9

$$s(t) = A_c(1 + k_a m(t))\cos(\omega_c t)$$

$$s(t) \times \cos(\omega_e t) = A_c(1 + k_a m(t))\cos(\omega_c t)\cos(\omega_e t)$$

$$= \frac{1}{2}A_c(1 + k_a m(t))\left[\cos(\omega_c - \omega_e)t + \cos(\omega_c + \omega_e)t\right]$$

We need $\;_c - \;_e = 455\,KHz$

$$\Rightarrow f_e = f_c - 455 = 1255 - 455$$

$$= 800\,KHz$$

Problem #10

For an FM signal: $\omega_i(t) = \omega_c + 2\pi k_f m(t)$

In our Problem: $\omega_i(t) = \dfrac{1}{\sqrt{LC}}$; with

$C = C_0(1 - Km(t))$.

Hence, $$\omega_i(t) = \frac{1}{\sqrt{LC_0(1 - Km(t))}}$$

$$= \frac{(1 - Km(t))^{-1/2}}{\sqrt{LC_0}}$$

$$\Rightarrow \omega_i(t) \cong \frac{1+\frac{1}{2}Km(t)}{\sqrt{LC_0}}; \text{ since } Km(t) \ll 1.$$

$$= \frac{1}{\sqrt{LC_0}} + \frac{Km(t)}{2\sqrt{LC_0}}$$

$$= \omega_c + 2\pi k_f m(t).$$

Hence, $2\pi k_f m(t) = \dfrac{Km(t)}{2\sqrt{LC_0}}$

$$\Rightarrow k_f = \frac{K}{4\pi\sqrt{LC_0}}$$

Problem #11

Determine first the modulation index, β.

$$\beta = \frac{2\pi k_f \max m(t)}{W} = \frac{k_f \max m(t)}{B}$$

$$= \frac{4688 \times 8}{15000} = 2.5$$

$$B_T = 2\ \ (\beta+1) = 2 \times 15(2.5+1)$$

$$= 105 \text{ KHz}.$$