EEC E442 Quiz #1 Solution Spring 2004-0

Problem #1

$S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$

$P_r\{s_i\} = P_i, \; i = 1, 2, 3, 4, 5, 6.$

(a) $A = \{s_1, s_2, s_3\}$

$B = \{s_4\}$

$C = \{s_5, s_6\}.$

(b) $P_r(A) = P_r(\{s_1\} \cup \{s_2\} \cup \{s_3\})$

$= P_r\{s_1\} + P_r\{s_2\} + P_r\{s_3\}$ since, $\{s_1\}, \{s_2\}, \{s_3\}$ are mutually exclusive events.

$= P_1 + P_2 + P_3$

$P_r(B) = P_r\{s_4\} = P_4$

$P_r(C) = P_r(\{s_5\} \cup \{s_6\}) = P_r\{s_5\} + P_r\{s_6\}$

$= P_5 + P_6$

(c) $P_r(A) + P_r(B) + P_r(C) = P_1 + P_2 + P_3 + P_4 + P_5 + P_6$

$= 1$

Since $A \cup B \cup C = S$ and A, B, C are mutually exclusive events. Hence,

$P_r\{(A \cup B \cup C)\} = P_r(A) + P_r(B) + P_r(C) = P(S) = 1.$

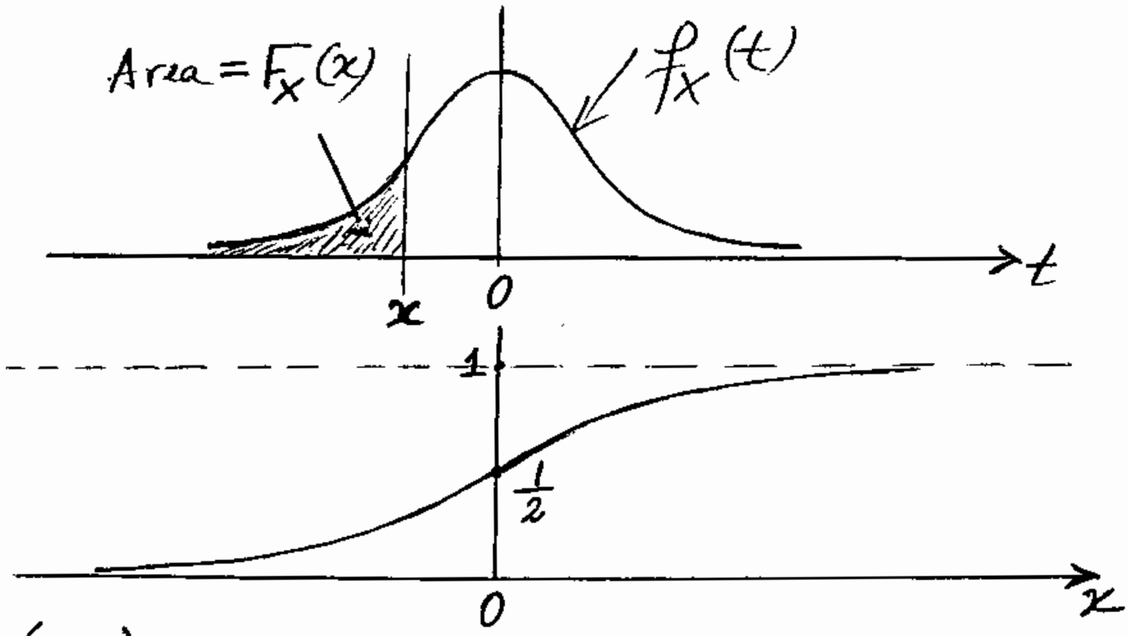
You can also use the fact that the elementary events are mutually and their union is equal to S.

Problem # 2

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}, \quad -\infty < x < \infty$$

(a)
$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$
$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}t^2\right\} dt$$

(b)



$F_X(-\infty) = 0,$

$F_X(0) = \frac{1}{2}$

$F_X(\infty) = 1$

Problem # 3

$$R_X(\tau) = E[X(t+\tau)X(t)]$$
$$= \exp\{-a|\tau|\}, \quad a > 0$$
$$-\infty < \tau < \infty$$

(a)

$R_X(\tau)$ as above holds for any t.
Hence it represents the correlation between any two samples of $X(t)$ with time difference τ.

The time difference τ can be positive or negative. As $|\tau|$ increases; i.e., the samples get farther away from each other, the correlation $R_X(\tau)$ decreases. Since $R_X(\tau) = e^{-a|\tau|}$, with $a > 0$.

(b) The process has a zero mean:

$$K_X(\tau) = R_X(\tau) - m_X^2 = R_X(\tau).$$

Hence, the samples are uncorrelated; $K_X(\tau) = 0$, if $R_X(\tau) = 0$. $\Rightarrow$ When $|\tau| \to \infty$ or $\tau \to \pm\infty$ the samples become uncorrelated.

(c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega = R_X(0)$ average power of $X(t)$.

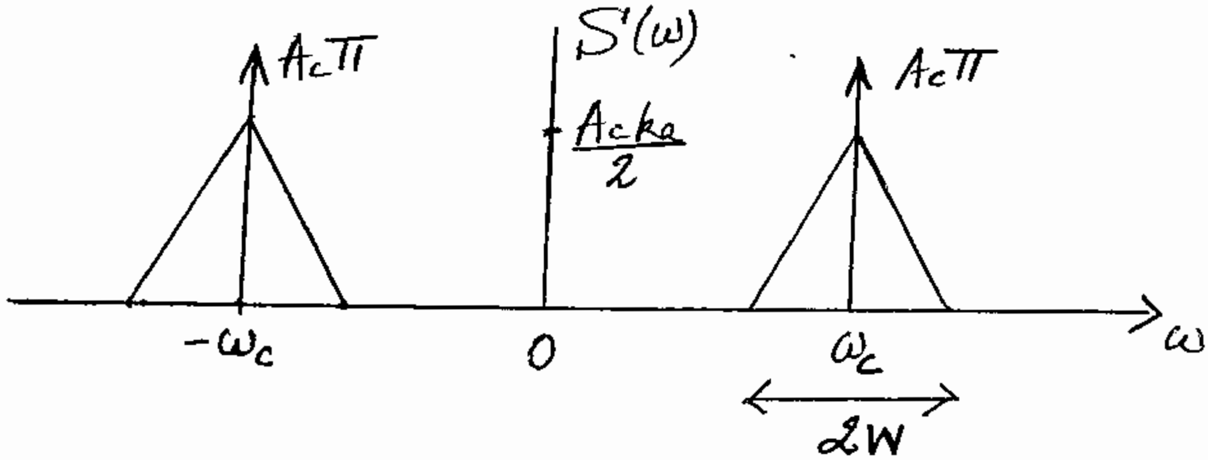
$$\Rightarrow \int_{-\infty}^{\infty} S_X(\omega)\, d\omega = 2\pi R_X(0) = 2\pi.$$

<u>Problem # 4</u> $s(t) = A_c \cos(\omega_c t) + A_c k_a m(t) \sin(\omega_c t)$

(a)
$$S(\omega) = A_c \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right] + \frac{1}{2\pi} A_c k_a M(\omega) * \frac{\pi}{j}\left[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)\right]$$

$$= A_c \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right] - \frac{j}{2} A_c k_a \left[M(\omega - \omega_c) - M(\omega + \omega_c)\right]$$

$$S(\omega) = A_c \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]$$
$$+ \frac{A_c k_a}{2} e^{-j\frac{\pi}{2}} M(\omega-\omega_c) + \frac{A_c k_a}{2} e^{j\pi/2} M(\omega+\omega_c)$$



(b) The transmission bandwidth of $s(t)$ is $2W$.

(c) $$s(t)\sin\omega_c t = A_c \cos\omega_c t \sin\omega_c t + A_c k_a m(t) \sin^2\omega_c t$$
$$= \frac{1}{2} A_c \sin(2\omega_c t) + \frac{1}{2} A_c k_a m(t) - \frac{1}{2} A_c k_a m(t)\cos(2\omega_c t)$$

A Low pass filter of bandwidth W picks $\frac{1}{2} A_c k_a m(t)$. Hence, a product modulator followed by a LPF can demodulate $s(t)$.

Problem # 5

The demodulation of a DSB-SC signal requires the use of a product modulator, LPF and synchronization circuitry. This is more complex than the simple use of an envelope detector followed by a LPF and a blocking capacitor. Hence, a DSB-LC receiver is simpler and cheaper. This justifies its use commercially.

Problem # 6

(a) $B_T = 2W(1+\beta) = 2 \times 52(1+2) = 312\,KHz$

Hence, 312 KHz is the least separation between the transmission frequencies of adjacent FM signals based on the FDM principle.

(b) The frequency multiplication factor is:

$n = \frac{2}{0.1} = 20.$

Problem # 7

(a) The envelope detector input is:

$$s(t)+n(t) = A_c\, m(t)\sin\omega_c t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

The envelope detector output is:

$$y_0 = \left\{ (A_c\, m(t) - n_s(t))^2 + n_c^2(t) \right\}^{1/2}$$

$$= \left\{ A_c^2 m^2(t) - 2A_c m(t) n_s(t) + n_c^2(t) + n_s^2(t) \right\}^{1/2}$$

$$= \left\{ A_c^2 m^2(t) \left[1 - \frac{2n_s(t)}{A_c m(t)} + \cancel{\frac{n_c^2(t) + n_s^2(t)}{A_c^2 m^2(t)}} \right] \right\}^{1/2}$$

$$= \left\{ A_c^2 m^2(t) \left[1 - \frac{2n_s(t)}{A_c m(t)} \right] \right\}^{1/2}$$ large carrier-to-noise ratio.

$$= A_c |m(t)| \left[1 - \frac{n_s(t)}{A_c m(t)} \right]$$

$$= \underbrace{A_c |m(t)|}_{\text{Signal Component}} \pm \underbrace{n_s(t)}_{\text{noise Component.}}$$

(b) $s(t) = A_c\, m(t)\cos\omega_c t.$

$$\Rightarrow y_0 = \left\{ (A_c\, m(t) + n_c(t))^2 + n_s^2(t) \right\}^{1/2}$$

$$= A_c |m(t)| \pm n_c(t).$$

(c) $$(SNR)_0 = \frac{A_c^2 P}{\frac{N_0 W}{\pi}}$$